ALAIN BADIOU'S MISTAKE — TWO POSTULATES OF DIALECTIC MATERIALISM

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Abstract. To accompany recent openings in category theory and philosophy, I discuss how Alain Badiou attempts to rephrase his dialectic philosophy in topos-theoretic terms. Topos theory bridges the problems emerging in set-theoretic language by a categorical approach that reinscribes set-theoretic language in a categorical framework. Badiou's own topos-theoretic formalism, however, turns out to be confined only to a limited, set-theoretically bounded branch of locales. This results with his mathematically reduced understanding of the 'postulate of materialism' constitutive to his account. Badiou falsely assumes this postulate to be singular whereas topos theory reveals its two-sided nature whose synthesis emerges only as a result of a (quasi-)split structure of truth. Badiou thus struggles with his own mathematical argument. I accomplish a correct version of his proof the sets defined over such a 'transcendental algebra' T form a (local) topos. Finally, I discuss the philosophical implications Badiou's mathematical inadequacies entail.

Contents

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Introduction

Alain Badiuo, perhaps the most prominent French philosopher today, is one of the key figures in *dialectic materialism* which is an increasingly popular discourse among continental philosophers and social scientists. It recourses to idealist philosophy in order to reassure the problems of contemporary scientific materialism. The *Being and Event* — which made Badiou popular since its publication in 1988 — introduced Cohen's mathematical procedure through which he established the independence of the continuum hypothesis. It served as a reflection of Badiou's paradoxic philosophy of the event. Badiou's ontology draws from the paradoxes and imperfection of set-theoretic formalism. The *Logic of Worlds*, published in 2006, made a shift towards a more phenomenologically oriented discussion on the present state of materialism. His 'postulate of materialism' signifies a 'real synthesis' in which the objectivity of the world emerges. Whilst he attempts to convey topos theory to support his argument, this postulate, as I demonstrate, entails two mathematically distinct conditions whose distinction Badiou disregards. Thus it falsifies Badiou's unilateral 'analytics' of such a 'real synthesis' the postulate entails. His account on 'democratic materialism' thus lies on mathematically false premises that pertain to his *locally* reductive account of topos theory. It relies solely on *locales* whose atomic implications do not subsist with general topos theory. Elementary topos theory, in contrast, gives a mathematically more pertinent grasp of what dialectic materialism theoretically entails.

Category theory is a structurally oriented foundation of mathematics alternative to set theory. Topos theory is a pivotal example of category theory's contribution to mathematics: an increasing number of mathematical domains have now been translated into topos-theoretic language whose internal semantics allows more conceptual forms of reasoning and the development of altogether new theorems in fields ranging from geometric quantum physics to probability theory. Category theory was first introduced by Samuel Eilenberg and Saunders Mac Lane during the 1940's. As a result, various fields of mathematics including algebraic geometry, geometric representation theory etc. have shifted away from previously dominant set-theoretic formalism. However, philosophers have paid more attention to the category-theoretic shift of the mathematical paradigm only recently[1].

Topos theory is a specific branch of categorically oriented mathematics which expresses logical, set-theoretic language as a synthetic discourse rather than an analytic condition of truth. It originated from a categorical representation of sheaf theory that gave rise to a new insight into set-theory. As such a counter-paradigmatic 'freebie', elementary topos theory was first designated by William Lawvere and Myles Tierney during the 1970's. They transferred Cohen's strategy to the algebrico-geometric sheaf-theoretic language employed by Alexander Grothendieck. It was a follow-up of Paul J.

[1] See Kömer, Ralf (2007), *Tool and Object: A History and Philosophy of Category Theory.* Science Networks. Historical Studies 32. Berlin: Birkhääuser.

Cohen's[2] original proof of the independence of the continuum hypothesis he had expressed during the 1960's.

Cohen had shown that set-theory involves sentences that are necessarily undecidable as one may construct two denumerable transitive model that give rise to two contradictory sentences. A topos, in contrast, effectuates a similar 'internal' (Mitchell–Bénabou-)language in diagrammatic terms. By embedding the topos of $\mathscr{S}ets$ into a so called *Cohen topos* it was possible to induce similar results. Reflecting its 'nomadic' status, topos theory was a development not inclined by the school of logic directly; Cohen himself was more involved in mathematical analysis than logic. As a mathematical double articulation, topos theory came to bridge set-theoretic formalism with category theoretic tools. Only later logicians themselves became interested in topos theory and the new, more 'structuralist' of mathematics category theory articulates. Rather than supposing the analytic primacy of set-theory as a 'meta-structural' foundation of mathematics in general, it made set-theoretic language an object of mathematical investigation.

For over a century, formal logic has dominated analytic philosophy which, after Bernard Russell, has strived for a solid foundation of thought and 'analytics'. More structurally oriented category theory grounds on a different formalism that questions the analytically primary foundations of set theory. In philosophical terms, category theory has been highlighted in connection to *structuralism*[3]. Structuralism is a paradigm emerging from structural linguistics developed by Ferdinand de Saussure. The paradigm was transferred to more social scientific contexts by such scholars as Claude Lévi-Strauss; it further became relevant to 'French mathematics' by the influence of Nicolas Bourbaki — a pseudonym of a mathematical society based in Paris. Despite its applications, category theory is still undermined by philosophical discourses preoccupied by the questions of formal logic and set theory. It thus seems not only an accident that topos theory as a structuralist approach to set-theoretic, 'analytic' problematics has now surfaced in France in fidelity to the 'French moment' of philosophy, as Badiou[4] regards it, which follows

[2]Cohen, Paul J. (1963), 'The Independence of the Continuum Hypothesis', *Proc. Natl. Acad. Sci. USA* 50(6). pp. 1143–1148.
Cohen, Paul J. (1964), 'The Independence of the Continuum Hypothesis II', *Proc. Natl. Acad. Sci. USA* 51(1). pp. 105–110.

[3]See for example
Awodey, S. (1996), 'Structure in Mathematics and Logic: A Categorical Perspective'. *Philosophia Mathematica* 4 (3). pp. 209–237. doi: 10.1093/philmat/4.3.209.
Palmgren, E. (2009), 'Category theory and structuralism'. url: www2.math.uu.se/~palmgren/CTS-fulltext.pdf, accessed Jan 1st, 2013.
Shapiro, S. (1996), 'Mathematical structuralism'. *Philosophia Mathematica* 4(2), pp. 81–82.
Shapiro, S. (2005) 'Categories, structures, and the Frege-Hilbert controversy: The status of meta-mathematics'. *Philosophia Mathematica* 13(1), pp. 61–62.

[4]Badiou, Alain (2012), *The Adventure of French Philosophy.* Trans. Bruno Bosteels. New York: Verso.

Sartre's 'phenomenological ontology'. Topos theory, as a structuralist 'phenomenological ontology', infers several relevant consequences the dialectic philosophy of subjectification which philosophers can no longer omit.

Multiple philosophers and scientists have blamed 'French philosophy' for its metaphoric approach to mathematics. Badiou's dialectic genius lies in his attempt to overcome this gap inasmuch as he claims to master also the *formal* discourse of mathematic he engages with. Aiming at a formally rigorous rereading of continental philosophy, Badiou's work has been praised for a poetically exquisite, original way in combining philosophical problematics in connection to set-theoretic axiomatics. His dialectic notion of the event derives directly from Russell's paradox which questions the foundations of set-theoretic formalism.

Unfortunately, regardless of his set-theoretically educated stance, Badiou cannot maintain a similar formal purity when it comes to category theory and topos theory in particular. The *Logic of Worlds* fails to sustain this dialectically critical insight on mathematical formalism the *Being and Event* illuminated. Whilst the *Being and Event* manages to shuffle through many of the debates and obscurities involved in formal logic, the shift towards the categorically oriented outlook of in the *Logic of Worlds* fails to maintain his stance. As a result, he seems to contradict his own 'meta-ontological thesis' that philosophers 'must study the mathematicians of their time'.

My question concerns precisely how well, and how rigorously in respect to formal topos theory Badiou succeeds in his ambitious project. If not, what are the philosophical implications of his occasional failures? Ironically, Badiou[5] himself claims that '[i]f one is willing to bolster one's confidence in the mathematics of objectivity, it is possible to take even further the thinking of the logico-ontological, of the chiasmus between the mathematics of being and the logic of appearing'. It is my mission to take this chiasmus a step further.

Sections 1–5 concentrate on Badiou's own formalism. Sections 6–11 shift to a categorically oriented setting and express a correct version of the statements Badiou makes. This follows with a more philosophically oriented discussion of the implications of Badiou's mathematical inadequacies. The main works I mathematically follow are by Peter Johnstone[6] and Saunders MacLane and Ieke Moerdijk[7].

[5]Badiou, Alain (2009). *Logics of Worlds. Being and Event, 2.* Transl. Alberto Toscano. London and New York: Continuum. [Originally published in 2006.] p. 197.

[6]Johnstone, Peter T. (1977), *Topos Theory.* London: Academic Press.
Johnstone, Peter T. (2002). *Sketches of an Elephant. A Topos Theory Compendium.* Volume 1. Oxford: Clarendon Press.

[7] Mac Lane, Saunders & Ieke Moerdijk (1992), *Sheaves in Geometry and Logic. A First Introduction to Topos Theory.* New York: Springer-Verlag.

1. From Ontological Algebra to 'Phenomenological Calculus'

Analytic tradition of philosophy, since Kant's[8] *Critique of Pure Reason*, has concentrated on the questions of *causation* — the rules of deduction — that appear to be given a condition of thought given *a priori* even if any practical case subscribes causality to the synthetic realm. Following this puritanist orientation of 'proper philosophy', it is Badiou's[9] maxim that 'mathematics *is* ontology — the science of being qua being' — despite the dialectic 'impasses of logic' due to Gödel, Tarski and others. In the *Being and Event*, the 'impasses' of 'Platonic' analytics compose the 'event' from similar pathologies that break the set-theoretic consistency in fidelity of Russell's paradox. In Badiou's polemic 'misuse' of set-theoretic formalism, an event is a self-belonging, 'reflexive' set '$e \in e$' whose 'paradoxic' designation gives rise to an anomalous mixture of the predicates $\subset$ and $\in$. In particular, the existence of such an 'event-multiple' obstructs the axiom of foundation.

1.1. **Axiom** (Foundation)**.** For each non-empty set x there is an element $y \in x$ so that their intersection $x \cap y = \varnothing$ is empty.

1.2. **Theorem.** *If there is a set e for which $e \in e$, then the axiom of foundation doesn't hold.*

Proof. If $e \in e$, then $y \in \{e\}$ implies $y = e$ so that one necessarily has $e \in y$ and $e \in \cap\{e\}$ which implies $e \in y \cap \{e\}$. This contradicts the axiom of foundation. □

Whilst the *Logic of Worlds* begins from a different problematics pertinent to 'phenomenological calculus' rather than 'meta-ontological caesura', the event-philosophy of the *Being and Event* already anticipates the topos theoretic shift which materialises these 'impasses' in alterantive, categorically founded terms. In effect, Badiou connects his dialectic notion of 'truth-process' — which replaces an event as its mathematical subjectification — with Cohen's strategy and the establishment of the independence of the continuum hypothesis (in particular, the establishment of contrary points of truth). Whilst nothing in Cohen's demonstration violates the axiom of foundation itself, Badiou 'metaphorically' associates the event-multiple e with a set ♀ which is 'supernumerary' to a particular transitive, denumerable model of set theory S. Although nothing in the process involves a *strictly logical* impasse, something during that transition $S \to S(♀)$ anyway changes; *something happens*.

But what precisely is this that happens in such a transitory passage between two models? That is the ultimate question which haunts Badiou in the *Logic of Worlds*. Rather than answering this question rigorously, Badiou's originality derives from his attempt to re-phrase the question of the event in

[8]Kant, Immanuel (1855), *Critique of Pure Reason.* Trans. J. M. D. Meiklejohn. London: Henry G. Bohn.

[9]Badiou, Alain (2006), *Being and Event.* Transl. O. Feltman. London, New York: Continuum. [Originally published in 1988.] p. 4

mathematically different, topos theoretic terms. His answer obviously fails. He says: nothing really happens, when mathematics is concerned. That is, he believes topos theory to provide no new insight to the event. But this answer follows from false premises: it derives only from his confined account of logically bounded, local topos theory. The only form of change it formally supports consists of 'modifications' that, when it comes to the causal order of logic, 'change nothing'. And Badiou is, in fact, right, but only insofar as mathematics is set-theoretically bounded, topos theory restricted to *local* Grothendieck-topoi.

What is crucial to an event is not how it is regulated but the way in which it 'makes the difference: not in space and time, but to space and time'[10]: Kant's *a priori* categories of thought. It is obvious that, insofar as space and time are represented in set-theoretic terms nothing radically happens. Badiou failst to answer 'what happened' except negatively. But his answer seems wrong: topos theory may say something interesting about (at least) a mathematical event. It invokes a better answer. Topos theory may say something interesting regarding the *changing concepts* of space and time — a real change in the formalism by which they are designated. Something in terms of Kant's 'trans-phenomenal real' has really changed!

Badiou shares Kant's idea of causation as an analytic condition of thought. But he misarticulates set theory itself as a purely analytic category — as a 'meta-structural' foundation of such 'dogmatic fidelity' of causation. What Badiou disregards is that ultimately Kant's categorical imperative, when topos theory synthetically applies it to the idealist logic itself, appears to withdraw the analytic status of the logical conditions of causation. This is because the categories of logic and the 'Platonic' ontology of $\mathscr{S}ets$ turn out to be more synthetic that Badiou conceives. Topos theory takes logic itself as its *object of synthesis* when it incarnates Cohen's strategic procedure topologically: space and time cannot be conceived in purely logical terms anymore.

Based on Badiou's earlier writings on category theory, Madarasz[11] argues that the way in which Badiou favors set theory, his 'transitory ontology', risks turning into a set-theoretic reduction of categorical orientation of mathematics. It is the same set-theoretic reduction of ontology which makes the *Logic of Worlds* mangle in the topos-theoretic drift sand. It misunderstands how ontology itself changes towards a more 'synthetically' oriented sphere of mathematics. This article shows how Badiou reduces topos theory to the theory of the so called *locales*: local Grothendieck-topoi that support generators, or equivalently, for which the geometric morphism $\gamma : \mathscr{E} \to \mathscr{S}ets$ is *bounded* and thus *logical.* In contrast, generally the 'internal' logic of a topos does not need to agree with the logic on 'external' Heyting algebra as Badiou's 'transitory cancellation' falsely presumes. This 'cancellation'

[10]Fraser, Mariam, Kember, Sarah & Lury, Celia (2005), 'Inventive Life. Approaches to the New Vitalism'. *Theory, Culture & Society* 22(1): 1–14.

[11]Madarasz, Norman (2005), 'On Alain Badiou's Treatment of Category Theory in View of a Transitory Ontology', In Gabriel Riera (ed), *Alain Badiou — Philosophy and its Conditions.* New York: University of New York Press. pp. 23–44.

forces such an agreement between two forms of logos explicitly and that is from where Badiou's 'real synthesis', the 'postulate of materialism', truly grounds.

2. Category Theory — A New Adventure in Philosophy?

By arguing that the formalism of the *Logic of Worlds* 'is very different from the one found in the *Being and Event*: from '*onto*-logy' to 'onto-*logy*', Badiou refers to category theory[12]. It is a different formalism which changes the *a priori* forms of sensibility; thus it constitutes a change in Kant's 'trans-phenomenal real'[13] as well. It gives rise to a designation of an 'object' approached relationally rather than absolutely as if to 'exparate' it by Kant's 'legalism—always asking *Quid juris?* or 'Haven't you crossed the limit?"[14] In this Kantian, categorical spirit, an object isn't defined in respect to what it consists and incorporates but in respect to how it relates to other objects in a more inscriptive setting. Category theory, contra Badiou's 'Platonically' reductive approach, respects Kant's 'sanctimonious declaration that we can have no knowledge of this or that'[15] as such an inquiry would be 'always threatening you with detention, the authorization to platonize'[16].

2.1. **Definition.** A category $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}(\mathcal{C})$ and a class (possibly not a set in the ZFC axiomatics) morphisms or arrows $\mathrm{Hom}(A, B)$ for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$. Furthermore, there is the associative operator $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ that connects any suitable pair of arrows.

What is crucial is that the objects themselves do not necessarily consist of points but instead they individuate with respect to their structural behaviour according to the morphisms that relate them together. These relations give rise to so called 'diagrams' that in turn are functorially transferred between different categories.

2.2. **Definition.** A (covariant) functor between categories $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a suitable set of maps $F : \mathrm{Ob}(\mathcal{C}_1) \to \mathrm{Ob}(\mathcal{C}_2)$ and $F : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ and it is a relation between two categories of compositions similarly as a function relates two different sets of consistencies.

Although the functor itself is defined as a function between *sets* of objects and morphisms in the case of a 'small category', contemporary mathematical practice takes the categorical relations *as primary* over the functional representation of point-sets. Badiou precisely fails to grasp this new 'ethos' of categorical mathematics. The *new*, 'Kantian' idea operates such relative identities in a structural manner: it approaches certain objects such as sheaves and the objects of which they are composed only insofar as they relate to each other. Badiou's reductive account, in contrast, deals with the

[12] Badiou, *Logics of Worlds*, 2009. p. 39.
[13] Ibid. p. 104.
[14] Ibid. p. 104.
[15] Ibid. p. 535.
[16] Ibid. p. 536.

category of T-sets alone and doesn't amount to such functorial descriptions or to a categorical understanding of points.

Instead, the categorical approach to points is based on the topological, *operative procedure* of localisation instead of designating themas the initial atoms of being as in the set-theoretic designation of Kuratowski-topology. As the *Being and Event* discusses, the set-theoretic foundations are based on taking the set consisting only of the void $\{\varnothing\}$ as the initial unit. In category theory, in contrast, objects could generally be regarded as 'void' precisely as they do not consist of anything; because, according to Kant's limited maxim, one cannot know what they 'consist of' and should not cross the limit of their direct apprehension. All the objects are thus 'atomic' in the sense of consistence but they are 'atoms' that can retain various relations including non-trivial internal morphisms, that is, *non-trivial self-relations* — endomorphisms. Only Badiou's 'Platonic', set-theoretically indoctrinated designation of points fails to grasp such 'intra-atomic' relations.

The new, modally different approach to objects and points reflects a transformation of mathematical objectivity towards more 'structuralist' direction. Badiou's own passage between the *Being and Event* and the *Logic of Worlds* exemplifies such a transformation at least metaphorically, whilst his formal passage from set theory to the theory of locales fails to count as a 'transphenomenally' real change. It is true that Badiou attempts to reformulate his approach in categorical, *functorial* terms. But this is a project he fails to comprehend as I demonstrate on the basis of his own formalism below.

3. Badiou's Ontological Reduction

To Badiou, 'mathematics', in particular set theory, 'is ontology' and thus the meta-structural framework which grounds his philosophy. Before elaborating category theory and *topos theory* further, let me review what Badiou himself formally accomplishes. This involves the process in which Badiou begins by defining an *external* complete Heyting algebra T and then demonstrating that the category of the so called T-sets[17] — 'locales' that are in fact also 'sets' in the traditional sense of set theory — is really a category in the sense of general category theory. Insofar as his theory of T-sets is constitutive to his theory of the worlds, this 'constitution' isn't mathematically necessary but relies only on *Badiou's own decision* to work on this particular regime of objects instead of category theoretically oriented topos theory in general. This problematics becomes particularly visible in the designation of the world $\mathbf{m}$ as 'a 'complete' (presentative) situation of being of '*universe* [which is] the (empty) concept of a being of the Whole'[18]. He recognises the 'impostrous' nature of such a 'whole' in terms of Russell's paradox, but in actual mathematical practice the 'whole' $\mathbf{m}$ becomes to signify the category

[17]In mathematical logic these structures are usually called Ω-sets, but I try to avoid a confusion between the internal, categorical object Ω and the extensive grading of a locale resulting from a push-forward $T = \gamma_*(\Omega)$ in a bounded morphism $\gamma : \mathcal{C} \to \mathscr{S}ets$.

[18]Badiou, *Logics of Worlds*, 2009. pp. 102, 153–155.

of $\mathscr{S}ets$[19] if one is to equip Badiou's accomplishments any formally sensible interpretation.

3.1. **Definition.** An external Heyting algebra is a set T with a partial order relation $<$, a minimal element $\mu \in T$, a maximal element $M \in T$. It further has a 'conjunction' operator $\wedge : T \times T \to T$ so that $p \wedge q \leq p$ and $p \wedge q = q \wedge p$. Furthermore, there is a *proposition* entailing the equivalence $p \leq q$ if and only if $p \wedge q = p$. Furthermore $p \wedge M = p$ and $\mu \wedge p = \mu$ for any $p \in T$.

3.2. *Remark.* In the language diagrammatically '*internal*' to a topos, the minimal and maximal elements of the lattice Ω can only be presented diagrammatically. The internal order relation $\leq_\Omega$ can be defined as the so called *equaliser* of the conjunction $\wedge$ and projection-map

$$\leq_\Omega \xrightarrow{e} \Omega \times \Omega \xrightarrow[\pi_1]{\wedge} L.$$

The symmetry can be expressed diagrammatically by saying that

$$\begin{tikzcd}
\leq_\Omega \cap \geq_\Omega \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook, "\Delta \circ \iota"] & \leq_\Omega \arrow[d, hook, "e"] \\
\geq_\Omega \arrow[r, hook] & \Omega \times \Omega
\end{tikzcd}$$

is a pull-back and commutes. The minimal and maximal elements, in categorical language, refer to the elements evoked by the so called *initial* and *terminal* objects 0 and 1.

In the case of *local Grothendieck-topoi* — Grothendieck-topoi that support generators — the external Heyting algebra T emerges as a push-forward of the internal algebra Ω, the logic of the external algebra $T := \gamma_*(\Omega)$ is an analogous push-forward of the internal logic of Ω but this is not the case in general.

What Badiou further requires of this 'transcendental algebra' T is that it is *complete* as a Heyting algebra.

3.3. **Definition.** A *complete* external Heyting algebra T is an external Heyting algebra together with a function $\Sigma : \mathbf{P}T \to T$ (the least upper boundary) which is distributive with respect to $\wedge$. Formally this means that $\Sigma A \wedge b = \Sigma\{a \wedge b \mid a \in A\}$.

3.4. *Remark.* In terms of the subobject classifier Ω, the envelope can be defined as the map $\Omega^t : \Omega^\Omega \to \Omega^1 \cong \Omega$, which is internally left adjoint to the map $\downarrow seg : \Omega \to \Omega^\Omega$ which takes $p \in \Omega$ to the characteristic map of $\downarrow(p) = \{q \in \Omega \mid q \leq p\}$[20].

The importance the external complete Heyting algebra plays in the intuitionist logic relates to the fact that one may now define precisely such an intuitionist logic on the basis of the operations defined above.

[19] See p. 28, ft. 67.

[20] Johnstone, *Topos Theory*, 1977, pp. 147–148.

3.5. **Definition** (Deduction)**.** The dependence relation $\Rightarrow$ is an operator satisfying

$$p \Rightarrow q = \Sigma\{t \mid p \cap t \leq q\}.$$

3.6. **Definition** (Negation)**.** A negation $\neg : T \to T$ is a function so that

$$\neg p = \Sigma\{q \mid p \cap q = \mu\},$$

and it then satisfies $p \wedge \neg p = \mu$.

Unlike in the case which Badiou[21] phrases as a 'classical world' (usually called a Boolean topos), where $\neg\neg = 1_\Omega$, the negation $\neg$ does not have to be reversible in general. In the case of locales, this is only the case when the so called *internal axiom of choice*[22] is valid, that is, when epimorphisms split as in the case of set theory. However, one always has $p \leq \neg\neg p$.

4. Atomic Objects

Badiou criticises the proper form of intuition associated with such multiplicities as space and time given their obscurity. However, his own mathematical 'intuition' grounds anchors to set theory as a basis of being. He claims this to provide him an access to what goes beyond the mere 'appearance' of objects limited '*Quid juris?*'. Instead of proceeding beyond his own ontologically 'transitory' intuition, Badiou falsely argues to grasp 'objects' in general, whilst, in fact, he discusses only particular type of objects: 'atomic' T-sets. If topos theory designates the subobject-classifier Ω relationally, the external, set-theoretic T-form reduces the question of truth again into such incorporeal framework split by an explicit order-structure $(T, <)$ or even $\{\text{false} < \text{true}\}$ contra the more abstract relation $1 \to \Omega$ pertinent to categorically abstract designation of topos theory. This *set-theoretically forced incorporation* of *classification and truth* is detrimental not only to Badiou's understanding of topos theory but also to his seek to grasp the question of unity relevant to the categorical notion of the object. As Badiou[23] regards this categorical unity, it is '[t]he transcendental unity of apperception is that unity through which all the manifold given in an intuition is united in a concept of the object'. Rather than seeing 'beyond' this universal property of 1 as a terminal object in a category, Badiou reductively negates this position when his 'analytics' beings from the atomic divisibility as a primary condition given '*a priori*' and effectuates such a *categorical synthesis of the object only secondarily.*

Indeed, Badiou's 'analytics' begins with a 'transcendental grading' of the world according to the external Heyting algebra T. In other words, he begins with a mathematical entity $(A, \mathbf{Id})$ where A is a set and $\mathbf{Id} : A \to T$ is a function satisfying specific conditions.

4.1. **Definition** (Equaliser)**.** First, there is an 'equaliser' to which Badiou refers as the 'identity' $\mathbf{Id} : A \times A \to T$ satisfies two conditions:

[21] Badiou, *Logics of Worlds*, 2009. pp. 183–188.
[22] Eg. Johnstone, *Topos Theory*, 1977, 141.
[23] Badiou, *Logics of Worlds*, 2009. p. 231.

1) symmetry: $\mathbf{Id}(x, y) = \mathbf{Id}(y, x)$ and
2) transitivity: $\mathbf{Id}(x, y) \wedge \mathbf{Id}(y, z) \leq \mathbf{Id}(x, z)$.

They guarantee that the resulting 'quasi-object' is objective in the sense of being distinguished from the gaze of the 'subject': 'the differences in degree of appearance are not prescribed by the exteriority of the gaze'[24].

4.2. *Remark.* This analogous 'identity'-function actually relates to the structural equalisation as it appears in category-theoretic language. Identities can be structurally understood as equivalence-relations. Given two arrows $X \rightrightarrows Y$, an *equaliser* (which always exists in a topos, given the existence of the subobject classifier Ω) is an object $Z \to X$ such that both induced maps $Z \to Y$ are the same. Given a topos-theoretic object X and U, pairs of elements of X over U can be compared or 'equivalised' by a morphism $X^U \times X^U \xrightarrow{eq} \Omega^U$ sturcturally 'internalising' the synthetic notion of 'equality' between two U-elements.[25]

Now it is possible to formulate the cumbersome notion of the 'atom of appearing' together with the formalism of such T-sets $(A, \mathbf{Id})$.

4.3. **Definition.** An *atom* is a function $a : A \to T$ defined on a T-set $(A, \mathbf{Id})$ so that

(A1) $a(x) \wedge \mathbf{Id}(x, y) \leq a(y)$ and
(A2) $a(x) \wedge a(y) \leq \mathbf{Id}(x, y)$.

In Badiou's vocabulary, an atom can be defined as an '*object-component which, intuitively, has at most one element in the following sense: if there is an element of A about which it can be said that it belongs absolutely to the component, then there is only one. This means that every other element that belongs to the component absolutely is identical, within appearing, to the first*'[26].

4.4. *Remark.* These two properties in the definition of an atom is highly motivated by the theory of T-sets (or Ω-sets in the standard terminology of topological logic). A map $A \to T$ satisfying the first inequality is usually thought as a 'subobject' of A, or formally a T-subset of A. The idea is that, given a T-subset $B \subset A$, I can consider the function

$$\mathbf{Id}_B(x) := a(x) = \Sigma\{\mathbf{Id}(x, y) \mid y \in B\}$$

and it is easy to verify that the first condition is satisfied. In the opposite direction, for a map a satisfying the first condition, the subset

$$B = \{x \mid a(x) = \mathbf{E}x := \mathbf{Id}(x, x)\}$$

is clearly a T-subset of A. The second condition states that the subobject $a : A \to T$ is a *singleton.* This concept emanates from the topos-theoretic internalisation of the singleton-function $\{\cdot\} : a \mapsto \{a\}$ which determines a particular class of T-subsets of A that correspond to the atomic T-subsets.

[24]Ibid., 205.
[25]Johnstone, *Topos Theory*, 1977, pp. 39–40.
[26]Badiou, *Logics of Worlds*, 2009. p. 248.

For example, in the case of an ordinary set S and an element $s \in S$ the singleton $\{s\} \subset S$ is a particular, atomic type of subset of S.

The dualist designation of elements distinguishes between two different sense for an element to be atomic: while 'the element depends solely on the pure (mathematical) thinking of the multiple', the second sense relates it 'to its transcendental indexing'[27]. In topos-theoretic register this interpretation is a bit more cumbersome[28].

Badiou still requires a further definition in order to state the 'postulate of materialism'.

4.5. **Definition.** An atom $a : A \to T$ is *real* if if there exists an element $x \in T$ so that $a(y) = \mathbf{Id}(x, y)$ for all $y \in A$.

This definition now gives rise to the postulate inherent to Badiou's understanding of 'democratic materialism'.

4.6. **Postulate** (Postulate of Materialism)**.** In a T-set $(A, \mathbf{Id})$, every atom of appearance is real.

4.7. *Remark.* What the postulate designates is that there really needs to *exist* $s \in A$ for every suitable subset that structurally (read categorically) *appears* to serve same relations as the singleton $\{s\}$. In other words, what 'appears' materially, according to the postulate, has to 'be' in the set-theoretic, incorporeal sense of 'ontology'. Topos theoretically this formulation relates to the so called *axiom of support generators* (SG), which states that the

[27]Ibid., 221.

[28] In purely categorical terms, an atom is an arrow $X \to \Omega$ mapping an element $U \to X$ into $U \to X \to \Omega$, thus an element of Ω^X. But arrows $X \to \Omega$, given that Ω is the sub-object classifier, correspond to sub-objects of X. This localic characterisation of a 'sub-object' is thus topos-theoretically justified. The 'element' corresponding with a singleton is exactly the monic arrow from the sub-object to the object. Only in Badiou's localic case the order relation $\leq$ extends to a partial order of the elements of A themselves, and a closed subobject turns out to be the one that is exactly generated by it smallest upper boundary, its envelope as the proof of the existence of the transcendental functor demonstrates. In other words, an atom in Badiou's sense is an atomic subobject of the topos of T-sets. In the diagrammatic terms of general topos theory it agrees with the so called singleton map $\{\} : X \to \Omega^X$, which is the exponential transpose of the characteristic map $X \times X \to \Omega$ of the diagonal $X \hookrightarrow X \times X$.

Expressed in terms of the so called internal Mitchell–Bénabou-language, one may translate the equality $x =_A y = \mathbf{Id}_A(x, y)$ into a different statement $x \in_A \{y\}$ which gives some insight into why such an 'atom' should be regarded as a singleton. Similarly, if a only satisfies the first axiom, the statement $x \in A_a$ would be semantically interpreted to retain the truth-value $a(x)$. Atoms relate to another interesting aspect: the atomic subobjects correspond to arrows $X \to \Omega$, which allows one to interpret Ω^X as the power-object. Functorially, $\mathbf{P} : \mathscr{E}^{op} \to \mathscr{E}$ takes the object X to $\mathbf{P}X = \Omega^X$ and an arrow $f : X \to Y$ maps to the exponential transpose $\mathbf{P}f : \Omega^Y \to \Omega^X$ of the composite

$$\Omega^Y \times X \xrightarrow{1 \times f} \Omega^Y \times Y \xrightarrow{ev} \Omega,$$

where ev is the counit of the exponential adjunction. The power-functor is sometimes axiomatised independently, but its existence follows from the axioms of finite limits and the subobject-classifier reflecting the fact that these two aspects are separate supposition regardless of their combined positing in the set-theoretic tradition.

terminal object 1 of the underlying topos is a generator. This means that the so called global elements, elements of the form $1 \to X$, are enough to determine any particular object X. Thus, it is this specific, *generative* condition of the *terminal unit* which reflects the non-evident and objectively insufficient characteristics of Badiouian determination of 'unity' of 'objects'. Rather than following Kant in not crossing the boundary, Badiou grounds this synthetic unity specifically in the 'quasi-split' atomic consistence of the objects that, as such an intervention, is the *opposite* of the synthetic category of unity. That is precisely where his confused philosophical 'analytics' most strikingly fails.

4.8. *Remark.* Even without assuming the postulate itself, that is, when considering a weaker category of T-sets not required to fulfill the postulate, the category of quasi-T-sets has a functor taking any quasi-T-set A into the corresponding quasi-T-set of singletons SA by $x \mapsto \{x\}$, where $SA \subset \mathbf{P}A$ and $\mathbf{P}A$ is the quasi-T-set of all quasi-T-subsets, that is, all maps $T \to A$ satisfying the first one of the two conditions of an atom designated by Badiou. It can then be shown that, in fact, SA itself is a *sheaf* whose all atoms are 'real' and which then is a proper T-set satisfying the 'postulate of materialism'. In fact, the category of T-$\mathscr{S}ets$ is equivalent to the category of T-sheaves $\mathscr{S}hvs(T, J)$[29]. In the language of T-sets, the 'postulate of materialism' thus comes down to designating an equality between A and its completed set of singletons SA, which I will effectively demonstrate in next section.

The particular objects Badiou discusses can now be defined as such quasi-T-sets whose all atoms are real; they give rise to what Badiou phrases as the 'ontological category par excellence'[30].

4.9. **Definition.** An object in the category of T-$\mathscr{S}ets$ is a pair $(A, \mathbf{Id})$ satisfying the above conditions so that every atom $a : A \to T$ is real.

Next, though not specifying this in the original text, attempts to show that such 'objects' indeed give rise to a mathematical category of T-$\mathscr{S}ets$.

5. Badiou's 'Subtle Scholium': T-sets are 'Sheaves'

By following established accounts[31] Badiou attempts to demonstrate that T-sets defined over an external complete Heyting algebra give rise to a Grothendieck-topos — a topos of sheaves of sets over a category. As T is a set, it can be made a required category by deciding its elements to be

[29] Wyler, Oswald (1991), *Lecture Notes on Topoi and Quasi-Topoi.* Singapore, New Jersey, London, Hong Kong: World Scientific. , 263.
[30] Badiou, *Logics of Worlds*, 2009. p. 221.
[31] For example
Bell, J. L. (1988), *Toposes and Local Set Theories: An Introduction.* Oxford: Oxford University Press.
Borceux, Francis (1994), *Handbook of Categorical Algebra. Basic Theory. Vol. I.* Cambridge: Cambridge University Press.
Goldblatt, Robert (1984), *The Categorical Analysis of Logic.* Mineola: Dover.
Wyler, Oswald (1991), *Lecture Notes on Topoi and Quasi-Topoi.* Singapore, New Jersey, London, Hong Kong: World Scientific.

its objects and the order relations between its elements the morphisms. He introduces the following notation.

5.1. **Definition.** The self-identity or existence in a T-set $(A, \mathbf{Id})$ is

$$\mathbf{E}x = \mathbf{Id}(x, x).$$

5.2. **Proposition.** *From the symmetry and transitivity of* $\mathbf{Id}$ *it follows that*[32]

$$\mathbf{Id}(x, y) \leq \mathbf{E}x \wedge \mathbf{E}y.$$

5.3. **Theorem.** *An object* $(A, \mathbf{Id})$ *whose every atom is real, is a sheaf.*

The demonstration that every object $(A, \mathbf{Id})$ is indeed a sheaf requires the definition of three operators: compatibility, order and localisation.

5.4. **Definition** (Localisation)**.** For an atom $a : A \to T$, a *localisation* $a \mathbin{\int} p$ on $p \in T$ is the atom which for each y establishes

$$(a \mathbin{\int} p)(y) = a(y) \wedge p.$$

The fact that the resulting $a \mathbin{\int} p : A \to T$ is an atom follows trivially from the fact that (the pull-back operator) $\wedge$ is compatible with the order-relation. Because of the 'postulate of materialism', this localised atom itself is represented by some element $x_p \in A$. Therefore, the localisation $x \mathbin{\int} p$ also makes sense: $\mathbf{Id}(x \mathbin{\int} p, y) = \mathbf{Id}(x, y) \wedge p$.

Two functions f and g may are *compatible* if they agree for every element of their common domain $f(x) = g(x)$. But when one only defines f and g as morphisms on objects (say X and Y) there needs to be a concept for determining the *localisations* $f|_{X \wedge Y} = g|_{X \wedge Y}$.

5.5. **Definition** (Compatibility)**.** In Badiou's formalism, two atoms are *compatible* if

$$a \ddagger b \quad \Longleftrightarrow \quad a \mathbin{\int} \mathbf{E}b = b \mathbin{\int} \mathbf{E}a.$$

5.6. **Proposition.** *I already declared that* $\mathbf{Id}(a, b) \leq \mathbf{E}a \wedge \mathbf{E}b$*; it is an easy consequence of the compatibility condition that if* $a \ddagger b$*, then* $\mathbf{E}a \wedge \mathbf{E}b \leq \mathbf{Id}(a, b)$ *and thus an equality between the two. This equality can be taken as a definition of compatibility.*

Sketch. The other implication entailed by the proposed, alternative definition has a bit lengthier proof. As a sketch, it needs first to be shown that $a \mathbin{\int} \mathbf{Id}(a, b) = b \mathbin{\int} \mathbf{Id}(a, b)$, and that the localisation is transitive in the sense that $(a \mathbin{\int} p) \mathbin{\int} q = a \mathbin{\int} (p \wedge q)$[33], that is, it is compatible with the order structure. Such compatibility is obviously required in general sheaf theory, but unlike in the restricted case of locales, Grothendieck-topoi relativise (by the notion of a sieve) the substantive assumption of the order-relation. Only in the case of *locales* such hierarchical sieve-structures not only appear as local, synthetic effects but are predetermined globally — ontologically by the poset-structure T. Finally, once demonstrated that $a \mathbin{\int} (\mathbf{E}a \wedge \mathbf{E}b) = a \mathbin{\int} \mathbf{E}b$, the fact that $a \ddagger b$ is an easy consequence[34]. □

[32] Badiou, *Logics of Worlds*, 2009. p. 247.
[33] Ibid., 271–272.
[34] Ibid. p. 273.

5.7. **Definition** (Order-Relation)**.** Formally one denotes $a \leq b$ if and only if $\mathbf{E}a = \mathbf{Id}(a, b)$. This relation occurs now on the level of the object A instead of the Heyting algebra T. It is again an easy demonstration that $a \leq b$ is equivalent to the condition that both $a \ddagger b$ and $\mathbf{E}a \leq \mathbf{E}b$. Furthermore, it is rather straightforward to show that the relation $\leq$ is reflexive, transitive, and anti-symmetric[35].

Proof of the Theorem 5.3. The proof of the sheaf-condition is now based on the equivalence of the following three conditions:

$$\begin{array}{ccll} a & = & b \wr \mathbf{E}a & \iff \\ a \ddagger b & \text{and} & \mathbf{E}a \leq \mathbf{E}b & \iff \\ \mathbf{E}a & = & \mathbf{Id}(a, b). & \end{array}$$

These may be established by showing that $\mathbf{E}a = \mathbf{Id}(a, b)$, if and only if $a = b \wr \mathbf{E}a$. The sufficiency of the latter condition amounts to first showing that $\mathbf{E}(a \wr p) = \mathbf{E}a \wedge p$[36].

To proceed with the proof, I need to connect the previous relations to the envelope Σ. First, it needs to be shown that if $b \ddagger b'$, then

$$b(x) \wedge b'(y) \leq \mathbf{Id}(x, y)$$

for all x, y. This follows easily from the previous discussion. The crucial part is now to show that the function

$$\pi(x) = \Sigma\{\mathbf{Id}(b, x) \mid b \in B\}$$

is an atom if the elements of B are compatible in pairs[37]. This is because it then retains a 'real' element which materialises such an atom. The first axiom (A1)

$$\mathbf{Id}(x, y) \wedge \pi(x) \leq \pi(y)$$

is straightforward[38]. Now $\pi(x) \wedge \pi(y) = \Sigma\{\mathbf{Id}(b, x) \wedge \mathbf{Id}(b', y)\}$ and by the previous $\mathbf{Id}(b, x) \wedge \mathbf{Id}(b', y) \leq \mathbf{Id}(x, y)$ so $\mathbf{Id}(x, y)$ is an upper boundary, but since the previous Σ is the least upper bound, I have $\pi(x) \wedge \pi(y) \leq \mathbf{Id}(x, y)$. Therefore π is an atom and I can denote by ϵ the the corresponding real element. Then it is possible to demonstrate that $\mathbf{E}\epsilon = \Sigma\{\mathbf{E}b \mid b \in B\}$. It follows that ϵ itself is actually the least upper bound of B: there exists a real synthesis of B[39].

Badiou characterises this 'transcendental functor of the object'[40], in other words this *sheaf*, as 'not exactly a function' as it associates rather than elements, 'subsets'. A sheaf can thus be expressed as a strata in which each neighborhood (transcendental degree) U becomes associated with the *set* of sections defined over U, usually denoted by $\mathcal{F}(U)$. Therefore, a sheaf is actually a *functor* $\mathscr{C}^{op} \to \mathscr{S}ets$, where $\mathscr{C}$ is a category. In Badiou's

[35] Ibid. p. 258.
[36] Ibid., 273–274.
[37] Ibid. p. 263.
[38] See ibid. p. 264.
[39] Ibid., 265–266.
[40] Ibid. p. 278.

restricted case $\mathscr{C}$ is determined to be the particular kind of category $\mathscr{C}_T$[41] deriving directly from the poset T. It results with a functor ${\mathscr{C}_T}^{op} \to \mathscr{S}ets$ in the following manner. Formally, for an object A, define $\mathcal{F}_A(p) = \{x \mid x \in A \text{ and } \mathbf{E}x = p\}$. If there is any $y \in \mathcal{F}_A(p)$ with $\mathbf{E}y = p$, then the equation $\mathbf{E}(y \wr q) = \mathbf{E}y \wedge q$ amounts to $\mathbf{E}(y \wr q) = p \wedge q$. If $q \leq p$ then $\mathbf{E}y \wr q = q$ giving rise to a commutative diagram:

$$\begin{tikzcd} p \arrow[r, "\mathcal{F}_A"] \arrow[d, "\leq"] & \mathcal{F}_A(p) \arrow[d, "\cdot \wr q"] \\ q \arrow[r, "\mathcal{F}_A"] & \mathcal{F}_A(q) \end{tikzcd}$$

which guarantees $\mathcal{F}_A$ to be a functor and thus a presheaf.

Finally, one needs to demonstrate the sheaf-condition, the 'real synthesis' in Badiou's terminology. The functor $J(p) = \{\Theta \mid \Sigma\Theta = p\}$ forms a 'basis' of a so called Grothendieck-topology on T (see the next section). Let me now consider such a basis Θ and a collection x_q of elements where $q \in \Theta \in J(p)$. it derives from an imaginary section x_p, the elements would be pairwise compatible. In such a case, let me assume that they satisfy the 'matching' condition $x_q \wr (q \wedge q') = x_{q'} \wr (q \wedge q')$, which implies that $x_q \ddagger x_{q'}$. One would thus like to find an element x_p, where $x_q = x_p \wr q$ for all $q \in \Theta$. But to demonstrate that the elements x_q commute in the diagram, they need to be shown to be pairwise compatible. One thus chooses x_p to be the envelope $\Sigma\{x_q \mid q \in \Theta\}$ and it clearly satisfies the condition. Namely, I just demonstrated that the envelope $\Sigma\{x_q\}$ localises to x_q for all q, that is,

$$\Sigma\{x_q\} \wr q = x_q, \qquad \forall q,$$

and that $\mathbf{E}\Sigma\{x_q\} = p$. The fact that it is unique then will do the trick. In the vocabulary of the next section, I have thus sketched Badiou's proof that objects are those of the topos $\mathscr{S}hvs(T, J)$ (see the following remark). $\square$

As a result, Badiou nearly succeeds in establishing the first part of his perverted project to show that T-sets form a topos (while claiming to work on topos theory more broadly). In other words, T-sets are 'capable of lending consistency to the multiple' and express sheaf as a set 'in the space of its appearing'[42].

5.8. *Remark.* As a final remark, what Badiou disregards in respect to the definition of an object, given a region $B \subset A$, whose elements are compatible in pairs, he demonstrates that the function $\pi(x) = \Sigma\{\mathbf{Id}(b, x) \mid b \in B\}$ is an atom of A. If one wants to consider the smallest $\bar{B}$ containing B within A which itself is an object, the 'real' element representing $\pi(x)$ should by the postulate of materialism itself lie in the atom: $\epsilon \in \bar{B}$ and thus because all elements are compatible, every element $b' \in \bar{B}$ has $b' \leq b$. Therefore, the

[41] Its objects are the elements $\mathrm{Ob}(\mathscr{C}_T) = \mathbf{P}T$. Now for $p, q \in T$ define $\mathrm{Hom}_{\mathscr{C}_T}(p, q) = \{\leq\}$ if $p \leq q$ and $\mathrm{Hom}_{\mathscr{C}_T}(p, q) = \varnothing$ otherwise.

[42] Badiou, *Logics of Worlds*, 2009. pp. 225–226.

sub-objects of the object A are generated by the ideals $\downarrow(\epsilon)$, each of them purely determined by the arrow $1 \to \{\epsilon\} \to \bar{B}$. (See remark 6.9.)

6. Categorical Approach and the Changing 'Materiality'

As discussed in the previous section, category theory is a general tendency of mathematics moving away from the question of content and consistence towards the problem of compositions — the art of not crossing the limit of what they absolutely 'consist of' but by approaching objects in more relative, categorical terms. As I argued, the transformation of mathematics during the latter half of the twentieth century seems to manifest relevant analogues to Kant's transcendental philosophy even if Badiou is reluctance to give up the the 'Platonist' perspective of set theory. In respect to the set-theoretic sphere of 'logical impostures' it is only a particular branch of category theory — topos theory — which becomes crucial to the 'Platonist' project as it combines the logic of consistencies with the new 'Kantian' language of compositions of objects. Topos theory — generally a theory of the categories of the so called *sheaves* — intersects the surface amidst set-theoretic problematics and categorical designation. In effect, one needs the theory of sheaves in order to effectuate the notion of 'space' in categorical language. A sheaf of functions (or arrows) is an object not directly related to a particular function but a class of functions with varying domains. It is a *functorial composition of different sections* in order to give rise to effects similar to those pertinent to functions defined in set-theoretic sense.

6.1. **Definition.** If X is a topological space, then for each open set U the sheaf $\mathcal{F}$ consists of an object $\mathcal{F}(U) \in \mathrm{Ob}(\mathcal{C})$ satisfying two conditions. First, there needs to exist certain 'natural' inclusions, so called restrictions

$$\mathcal{F}(U) \to \mathcal{F}(U') : f_U \mapsto f_U|_{U'}$$

for all open sets $U' \subset U$. There is also a *compatibility condition* which states that given a collection $f_i \in \mathcal{F}(U_i)$, there needs to be $f \in \mathcal{F}(\bigcup_i U_i)$ so that $f|_{U_i} = f_i$ for all i. In the categorical language, the sheaf-condition means that the diagram

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is a coequaliser for each covering sieve (U_i) of U, usually determined as the set $J(U)$ of such sieves.

6.2. *Remark.* The notion of a sheaf extends the 'point-wise' representation of a 'classical space' as a carrier of consistent functions by a more 'diagrammatic' idea composing the functorial relations behind such carriage. To exemplify this, let me assume I do not know anything about the thing X, whether it is a space or not. If someone tells me for any 'given' multiple U that there are arrows (relations) $X(U) = \{U \to X\}$, would I be able to guess how the thing X looks like from this information? Not quite. Such information is required but I, in addition, need to know how different ways of probing X via arrows $U \to X$ relate to each other. For every pair U and

V of test spaces that we come up with, and for every arrow $\varphi : U \to V$ mapping these into each other, I need further information how φ takes the elements of $X(V)$ into elements of $X(U)$ via the 'composition' $U \to V \to X$. The 'sheaf'-conditions turn out to be enough to guarantee that this information amounts to an essentially unique X, the object called a sheaf. It might the space X itself in which case the sheaf X would be *representable*; that is I would have $X(U) = \mathrm{Hom}(U, M)$ represented by the space M.

6.3. *Remark* (Representable sheaves). By the so called Yoneda lemma the transformation from 'classical spaces' to sheaves is natural in the sense that given any 'classical space' denoted as an object A, the natural transformations
$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "h_A"] \arrow[r, bend right, "\mathcal{F}"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{C} \end{tikzcd}$$
between the functors h_A and $\mathcal{F}$ corresponds to the elements of $\mathcal{F}(A)$. If 'spaces' are regarded themselves equated with their corresponding (pre)sheaves h_A, I then have $F(A) = \mathrm{Hom}(A, F)$. This gives as the famous Yoneda embedding

$$\mathbf{y} : \mathscr{C} \to \mathscr{S}ets^{\mathscr{C}^{op}},$$

which embeds the index-category $\mathscr{C}$ into its corresponding Grothendieck-topos $\mathscr{S}ets^{\mathscr{C}^{op}}$. It maps objects of $\mathscr{C}$ into *representable sheaves* in $\mathscr{S}ets^{\mathscr{C}^{op}}$.

6.4. *Remark.* The above definition is based on the Bourbakian, set-theoretic definition of topology as a collection of open subsets (X) (which is a locale). It gives rise to a similar structure Badiou defines as the 'transcendental functor', that is, a sheaf identified with the set-theoretically explicated *functional strata.* Topos theory makes an alternative, categorical definition of topology — for example the so called Grothendieck-topology. Regardless of the topological framework, the idea is to categorically impose such structures of localisation on $\mathscr{S}ets$ that make 'points', and thus the 'space' consisting of such points, to emerge. Whereas set theory presumes the initially atomic primacy of points, the sheaf-description of space does not assume the result of such localisation — the actual points — as analytically given but only synthetic result of the functorial procedure of localisation. However, inasmuch as sheaf-theory — and topos theory — aims to bridge the categorical register with the 'ontological' sphere of sets, such a strata of sets is still involved even in the case of topos theory. For a more general class of such categorical stratifications, one may move to the theory of grupoids and stacks, that is, categories fibred on grupoids.

6.5. **Definition** (Grothendieck-topology). Let $\mathscr{C}$ be a category. A *sieve* — in French a 'crible' — on C is a covering family of C so that it is downwards closed. A *Grothendieck-topology* on a category $\mathscr{C}$ is a function J assigning a collection $J(C)$ of sieves for every $C \in \mathscr{C}$ such that

1) the maximal sieve $\{f \mid \text{ any } f : D \to C\} \in J(\mathscr{C})$,
2) (stability) if $S \in J(C)$ then $h^*(S) \in J(D)$ for any $h : D \to C$, and
3) (transitivity) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all arrows $h : D \to C$, then $R \in J(C)$.

The stability condition specifies that the intersections of two sieves is also a sieve. Given the latter two conditions, a topology itself is a sheaf on the maximal topology consisting of all sieves. It is often useful to consider a basis; for example Badiou works on such a 'basis' in the previous proof without explicitly stating this.

6.6. **Definition.** A basis K of a Grothendieck-topology J consists of collections (not necessarily sieves) of arrows:

1) for any isomorphism $f : C' \to C$, $f \in K(C)$,
2) if $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, then the pull-backs along any arrow $g : D \to C$ are contained in $\{\pi_2 : C_i \times_C D \to D \mid i \in I\} \in K(D)$, and
3) if $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, and if for each $i \in I$ there is a family $\{g_{ij} : D_{ij} \to C_i \mid j \in I_i\} \in K(C_i)$, also the family of composites $\{f_i \circ g_{ij} : D_{ij} \to C \mid i \in I, j \in I_i\}$ lies in $K(C)$.

For such a basis K one may define a sieve $J_K(C) = \{S \mid S \supset R \in K(C)\}$ that generates a topology.

6.7. *Remark.* The notion of Grothendieck-topology enables one to generalise the notion of a sheaf to a broader class of categories instead of the (classical) category of Kuratowski-spaces. If **esp** is the category of topological spaces and continuous maps, sheaves on $X \in \mathrm{Ob}(\mathscr{C})$ are actually equivalent to the full subcategory of $\mathbf{esp}/(X, \mathscr{C})$ of local homeomorphism $p : E \to X$. The corresponding pair of adjoint functors $\mathscr{S}ets^{\mathscr{C}^{op}} \rightleftarrows \mathbf{esp}/(X, \mathscr{C})$ yields an associated sheaf functor $\mathscr{S}ets^{\mathscr{C}^{op}} \to \mathscr{S}hvs(X)$ which shows that sheaves can be regarded either as presheaves with exactness condition or as spaces with local homeomorphisms into X[43].

6.8. *Remark.* In the case of Grothendieck-topology, there is an equivalence of categories between the category of pre-sheaves $\mathscr{S}ets^{\mathscr{C}^{op}}$ and the category of sheaves $\mathscr{S}hvs(\mathscr{C}, J)$ where J is the so called canonical Grothendieck-topology. Thus one often omits the reference to particular topology and deals with presheaves $\mathscr{S}ets^{\mathscr{C}^{op}}$ instead, even if their objects do not satisfying the two sheaf-conditions.

6.9. *Remark.* Now I have introduced the formalism required to accomplish the final step of Badiou's proof in the previous section (see remark 5.8). Namely, the category $\mathscr{S}ets^{T^{op}}$ is generated by representable presheaves of the form $y(p) : q \mapsto \mathrm{Hom}_T(q, p)$ by the Yoneda lemma. Let

$$a : \mathscr{S}ets^{T^{op}} \to \mathscr{S}hvs(T, J)$$

[43] Johnstone, *Topos Theory*, 1977, pp. 10–11.

be the associated sheaf-functor $P \mapsto P^{++}$[44]. Then because T is a poset, for any $p \in T$, the map $\mathrm{Hom}(\cdot, p) \to 1$ is a mono, and because a is left-exact, also $ay(p) \to 1$ is a mono[45]. Through it, any sheaf is a subobject of 1. This is exactly the axiom of support generators (SG) that is crucial to Badiou's constitutive postulate of materialism: it follows exactly from the organisation of T as an ordered poset and this order-relation being functorially extendible to general objects of $\mathscr{S}hvs(T, J)$.

7. Historical Background: A Categorical Insight into Geometry

Grothendieck opened up a new regime of algebraic geometry by generalising the notion of a space first scheme-theoretically with sheaves and then in terms of grupoids and higher categories. Topos theory became synonymous to the study of categories that would satisfy the so called Giraud's axioms[46] based on Grothendieck's geometric machinery. By utilising such tools, Pierre Deligne was able to prove so called Weil conjecture, a mod-p analogue of the Riemann hypothesis. The conjectures — already anticipated in the work of Gauss — concern the so called local ζ-functions that derive from counting the number of points of an algebraic variety over a finite field, an algebraic structure similar to that of for example rational $\mathbb{Q}$ or real numbers $\mathbb{R}$ but with only a finite number of elements. Representing algebraic varieties in polynomial terms make it possible to analyse analogous geometric structures over finite fields $\mathbb{Z}/p\mathbb{Z}$, that is, the whole numbers modulo p. Whereas finite, 'discrete' varieties had previously been considered as rather separate from topology, now it occurred that geometry overcame such dualist topologically split distinction between 'continuous' and 'discrete'[47]. Such patterns as the Betti number reflected by the Weil conjectures weren't reducible to such a disparate dialectics anymore. It was almost as if the continuous and discrete presentations of geometry were just the two superficial faces of the much deeper geometric precursors involved. The concept of étale-cohomology generalised the notions of a sheaf and space turning local neighborhoods into algebraic morphisms. The topological question of *localisation* became a question of structural, categorical compositions instead of a direct, set-theoretic presentation of varieties. Thus Alexander Grothendieck

[44]The construction of the sheaf-functor proceeds by defining

$$P^+ = \varinjlim_{R \in J(C)} Match(R, P), \qquad P \mapsto P^{++},$$

where $Match(R, P)$ denotes the matching families. When applied twice, it amounts to a left-adjoint

$$a : \mathscr{S}ets^{\mathscr{C}^{op}} \to \mathscr{S}hvs(\mathscr{C}, J)$$

which sends a sheaf to itself in the category of presheaves.

[45]Mac Lane & Moerdijk, *Sheaves in Geometry*, 1992, 277.

[46]Johnstone, *Topos Theory*, 1977, p. xii.

[47]These are the two *distinct* 'modalities' of the subject as discussed in Badiou, *Logic of Worlds*, 2009. This again demonstrates how the account is set-theoretically reductionist in respect to the more abstract regime which doesn't make such a direct opposition.

revolutionarised the whole field of algebraic geometry on the basis of Jean-Pierre Serre's suggestions. This ultimately enabled Pierre Deligne to prove the result regarding the weights of the push-forwards of a sheaf which is even more general than the original Weil conjectures.

Grothendieck's crucial insight relied on his observation that if morphisms of varieties were considered by their 'adjoint' field of functions, the geometric epimorphisms related to monomorphisms of fields. There was a *structural analogue* between the consistent inclusions of spaces and the algebraic relations of fields. The analogue was not inevitable however: the category of fields did not have an operator analogous to pull-backs (fibre products) *unless embedded to a larger category of rings*, in which pull-backs have a co-dual of tensor products $\otimes$. While a traditional Kuratowski covering space is locally 'split' — as mathematicians call it — the same was not true for the dual category of fields in which every morphism is a mono, that is, an injection. This lead Grothendieck to understand the necessity to replace neighborhoods $U \hookrightarrow X$ with the more general category of maps $U \to X$ that are not necessarily monic.

Topos theory did not only rely on this geometric insight to replace 'split' morphisms, but proceeded in an other direction, which began from Lawvere's work on the categories of sheaves on $\mathscr{S}ets$. If the notion of a unit seemed to be a question of the beginning as expressed by set theory *initiating* from the 'void' ($\varnothing$), topos theory took replaced its concept of unity with a procedure of *termination*. That is, the unit 1 in a topos is the *terminal object* of the category. Ultimately Lawvere and Tierney saw the importance of *classification* in the new composed regime of category theory. Lawvere was investigating the structural emergence of 'points of truth' in such a 'relativist' category of objects. He regarded the set $\{\text{true}, \text{false}\}$ — in fact the so called *subobject-classifier* in the category of $\mathscr{S}ets$ — as the 'object of truth-values' in the category of $\mathscr{S}ets$ with a magnificient result: such an object in an arbitrary category enables us to reduce Axiom of Comprehension to a rather elementary statement regarding adjoint functors[48].

7.1. **Axiom** (Comprehension)**.** If λ is a property, then the set discerned by this property actually exists. In set-theoretical formalism it can be expressed as

$$\forall w_1, \dots, w_n \forall A \exists B \forall X (x \in B \iff [x \in A \wedge \lambda(x, w_1, \dots, w_n, A)]),$$

where $x, w_1, \dots, w_n$ are free variables of the statement λ.

Topological categories including elementary and Grothendieck-topoi retain such a truth(-value) object Ω, but it doesn't result with such a split, set-theoretic evaluation as in the case of locales graded according to the partially ordered values of T. The subobject-classifier, in contrast, is explicated by the *universal property* of the so called morphism true: $1 \to \Omega$ whose pull-back-property classifies subobjects of any subobject (mono) in that category (see definition 10.3). In particular, the endomorphisms of the

[48]Ibid., xiii.

truth-value object Ω are closely connected to Grothendieck-topology[49]. In general, however, the subobject-classifier doesn't need to set-theoretically split as in the set-theoretic case of $\Omega \cong \{\text{false} < \text{true}\}$ or in the case of locales in which $\Omega \cong T$.

As the last historical remark, topos theory became all the more important because of the so called Freyd–Mitchell embedding theorem for abelian categories. It guaranteed the explicit set of elementary axioms implying exactness properties of module categories as well as demonstrating how the abstract categorical approach would result with crucial module theoretic applications.

8. Postulate of Materialism — Or Two Postulates Instead?

The postulate of materialism dominates the Badiouian theory of 'democratic materialism' into which he aims to reduce such 'vitalist mysticists' as Bergson, Deleuze or even Leibniz. But in the functorial regime of category theory, there are actually two layers of assumptions involved even if they are invisible in the reduced, 'quasi-split' framework of T-$\mathscr{S}ets$. Thus Badiou's 'postulate of materialism' retains both a *weaker* and a *stronger* version when one moves beyond the functionalist definition of T-sets the compositional procedures in which the functionalist description is used. The weaker version merely states that objects of a topos are, in fact, sheaves of sets. It determines the class of the so called Grothendieck-topoi which, as a matter of fact, does not have much to do with the most original aspects of Alexander Grothendieck's work.

Badiou's objects may always be interpreted as Grothendieck-topoi (situated in $\mathscr{S}ets^{T^{op}}$) which is a 2-category distinct from the more abstractly designated 2-category elementary topoi ($\mathfrak{Top}$). An elementary topos $\mathscr{E}$ is a Grothendieck-topos only if there is a particular morphism $\mathscr{E} \to \mathscr{S}ets$ that 'materialises' its internal experience of truth even if this morphism wasn't 'bounded' in the sense that the internal logic of the topos $\mathscr{E}$ isn't reducible to the external logic of T-$\mathscr{S}ets$ (or $\mathscr{S}ets$).

8.1. **Postulate** (Weak Postulate of Materialism)**.** Categorically the weaker version of the postulate of materialism signifies precisely condition which makes an elementary topos a so called Grothendieck-topos (see remark 10.4).

I will now discuss how this weak postulate of materialism relates to what Badiou takes as the 'postulate of materialism'. A Groethendick-topos can generally be written as $\mathscr{S}ets^{\mathscr{C}^{op}}$ defined over any category $\mathscr{C}$ in which points might have internal automorphisms for example. This means that the 'sections' do *materialise* in the 'Platonic' category of $\mathscr{S}ets$ but its *internal structure* isn't reducible to this material *appearance*. In Badiou's case it is this this internal structure which becomes *bounded* by the apparent materialism. In effect, if the underlying 'index-category' $\mathscr{C} = \mathscr{C}_T$ a *poset* corresponding to the external Heyting algebra T, any internal 'torsion' such as non-trivial endomorphisms of $\mathscr{C}$ are extensively forced out.

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[49]Ibid., xiv.

What was crucial to Badiou's demonstration is that objects are atomic — every atom is 'real'. That is what I phrase as the *postulate of atomism* and in elementary topos theory it is often phrased as the axiom of support generators (SG). I will now overview what this means categorically. It accounts for the unity and singularity of the terminal object 1. In general, an object may be said to retain 'global elements' $1 \to X$, but there are also 'local elements' $U \to X$ that might not be determinable by global elements alone. For each arrow $U \to X$, a 'local' element on U, the pull-back object U^*X in the local topos $\mathscr{E}/U$ has exactly one global element of U^*X, that is, a local element of X over U[50]. The object X, as a whole, cannot be determined by 'global elements' alone, if there occurs torsion that obstructs such a direct localic hierarchy pertinent to the category of T-$\mathscr{S}ets$ — a poset-structure analogous to Cohen's posets of forcing which also grounds Badiou's '*fundamental law of the subject*'[51]. In other words, the geometric intuition of torsion becomes superfluous when obstructed in fidelity with the fundamental law of the subject on which Badiou's ontological understanding relies. For example, in the case of the Möbius strip, that is, the non-trivial $\mathbb{Z}/2\mathbb{Z}$-bundles (that have no global elements at all), there occurs such torsion which prevents an atomic representation of the object.

On the basis of the extent to which elements in a X are discerned by its global elements, one can now phrase the postulate of atomism which doesn't yet result with the complete 'postulate of materialism' as Badiou phrases it, however.

8.2. **Axiom** (Postulate of Atomism)**.** An elementary topos $\mathscr{E}$ *supports generators* (SG) if the subobjects of 1 in the topos $\mathscr{E}$ generate $\mathscr{E}$. This means that given any pair of arrows $f \neq g : X \rightrightarrows Y$, there is $U \hookrightarrow X$ such that the two induced maps $U \rightrightarrows Y$ do not agree. In the case of $\mathscr{E}$ being defined over the topos of $\mathscr{S}ets$, the axiom (SG) is equivalent to the condition, that the object 1 alone generates the topos $\mathscr{E}$[52]. If $\mathscr{E}$ satisfies (SG), then Ω is a cogenerator of $\mathscr{E}$ — differenciating between a parallel pair from the right — and for any topology j or an internal poset $\mathbf{P}$, both $\mathscr{E}^{\mathbf{P}}$ and $\mathscr{S}hvs_j(\mathscr{E})$ satisfy (SG)[53].

Based on the two postulates, the weak postulate of materialism and the postulate of atomism, it is now possible to draw the general and abstract *topos-theoretic conditions* of Badiou's 'postulate of materialism' which should rather be phrased as the *postulate of atomic materialism.*

8.3. **Postulate** (Strong Postulate of Materialism)**.** In addition to the weak postulate of materialism, the *strong postulate of materialism* presumes the Grothendieck-topos defined over $\mathscr{S}ets$ to additionally satisfy the postulate of atomism, that is, its objects are generated by the subobjects of the terminal object 1. Alternatively, this is the postulate of atomic materialism.

[50]Ibid., 39.
[51]Badiou, *Being and Event*, 2006, p. 401.
[52]Johnstone, *Topos Theory*, 1977, p. 145.
[53]Ibid., 146.

According to the following proposition, there is now a correspondence between topoi satisfying the strong postulate and locales, that is, the categories of T-$\mathscr{S}ets$ — namely, one chooses $T = \gamma_*(\Omega)$ for the morphism of *materialist appearance* γ.

8.4. **Proposition.** *If an elementary topos $\mathscr{E}$ is Grothendieck-topos defined over $\gamma : \mathscr{E} \to \mathscr{S}ets$, then it supports generators if and only if $\gamma : \mathscr{E} \to \mathscr{S}ets$ is* logical *and thus* bounded. *This means that the internal complete Heyting algebra Ω transforms into an external complete Heyting algebra $\gamma_*(\Omega)$ and then $\mathscr{E}$ is equivalent to the topos $\mathscr{S}ets^{\gamma_*(\Omega)^{op}}$.*

8.5. *Remark.* In general topos theory, the subobjects (monics) of the internal subobject-classifier Ω form a class of arrows $\mathrm{Hom}(\Omega, \Omega)$ as much as the global elements of Ω, that is, the arrows $1 \to \Omega$, corresponds to the subobjects of 1. For any object U there are also the subobjects $U^*\Omega$ corresponding to arrows $U \to \Omega$, so called *generalised points* whereas the localic theory of T-sets fails to resonate with such generalisation. Therefore, the designation of T as a set of global points is possible *only* for Grothendieck-topoi that support generators.

8.6. *Remark.* Obviously, if one works on an elementary topos $\mathscr{E}$ which is not defined over $\mathscr{S}ets$ (contrary to the case of Grothendieck-topoi), nothing guarantees that the axiom SG would imply the topos to be a locale. The implication is valid only when $\mathscr{E}$ is a Grothendieck-topos in the first place. The axiom SG could thus be taken as an alternative, 'weak' version of the 'postulate of materialism', whilst then the Grothendieck-condition, in addition, would make it strong. But because this alternative weak condition would then not entail such a 'materialisation' morphism $\gamma : \mathscr{E} \to \mathscr{S}ets$ to exists, nothing would make this postulate 'materialist'. Therefore one can rather safely articulate the two versions of the 'postulate of materialism' in the hierarchical order which they take in the regime of Grothendieck-topoi as long as one remembers to avoid such a hierarchical designation when discussing elementary topos theory — and the distinct postulates of atomism and weak materialism — in general.

9. The Inscriptive Frontier of Dialectical Materialism

The latter one of these assumptions, the *weak postulate of materialism*, is now the subject matter of this section. The strong, atomic version is constitutive to *democratic materialism* which, however, is a perspective that Badiou's dialectic ethos opposes. Such an atomic materialism prevents subjective torsion, which is in fidelity with Badiou's 'fundamental law of the subject'. In contrast, precisely by allowing torsion but regulating the object by categorical means, a Grothendieck-topos gives rise to an alternative notion of truth that should be taken as the basis for *dialectic materialism*[54].

[54]Reflecting its topicality, Slavoj Žižek has recently raised up the question of dialectic materialism in connection to multiple anomalies that have emerged in quantum physics. Žižek, Slavoj (2012), *Less Than Nothing. Hegel and the Shadow of Dialectical Materialism.* New York and London: Verso.

As I discussed before, sheaves relate to three operators: compatibility, order and localisation defined on the level of the object. In the case of a traditional (ontological) topological space X, a sheaf is just a presheaf $\mathcal{F} \in \mathscr{S}ets^{(X)}$ satisfying compatibility condition. This definition of a sheaf applicable to a locale (X) has, however, a flaw. Rather than there being an externally given structure of localisations (eg. (X)) *once and for all*, one needs to relativise different compositions of neighborhoods in which chains of localisation might occur as some implicit 'torsion' appears to obstruct their globalisation.

This Grothendieck's insight that didn't reject the possibility of such torsion in general whilst he managed to anyway provide mathematical machinery to deal with sites *material enough* for their synthetic regulation. Grothendieck-topologies became to provide such locally hierarchical 'snapshots' of the topology without supposing ontological flatness for the global 'whole'. Perhaps one could phrase that Grothendieck's register of *objectivity* was *synthetically local* rather than analytically global. A locale, that is a category of T-$\mathscr{S}ets$, in contrast, is a *local* Grothendieck-topoi precisely as it obstructs such global torsion in a predetermined fashion and is a hierarchical snapshot of the situation as a whole.

As discussed above, Grothendieck's strategy helps in multiple situations. For example in algebraic geometry on the traditional Zariski-site, there is not enough sub-objects to go by; the Kuratowskian topological category consisting of only inclusions is too small. In contrast, the 'generalised' neighborhoods $U \to X$ do not necessarily 'split'; they are not monic (read 'inclusions'); that is, they are inscribed rather than incorporeal neighbourhoods. For there to be such an injection that would require there to be a surjective algebraic morphisms of fields. But this is only possible in the case of an isomorphism. Therefore, there seems to be no good, incorporated neighbourhoods in the classical, incorporeal sense of Kuratowski-topology. The solution by *sieves* marked Grothendieck's trade-off. It results with particular coverings $U_i \to X$ with morphisms not necessarily split (non-monic) but where the index-category is still conceivable to approach by suitable 'partial' or 'local' filters (sieves). This retrospective indexing is a transcendental grading external to the 'ontological', set-theoretic structure of the multiple itself. That is precisely the inscriptive *Grothendieck-topology* J on a category $\mathscr{C}$, which associates a collection $J(C)$ of *sieves on* C to every object $C \in \mathscr{C}$, that is, downward closed covering families on C (see definition 6.5).

9.1. *Remark.* If $U \to X$ is in $J(C)$, then any arrow $V \to U \to X$ has to lie there as well. Thus, the association law makes sieve a *right ideal*. In general, if the functor $y : C \to \mathrm{Hom}(\cdot, C) : \mathscr{C} \to \mathscr{S}ets$ is considered as a presheaf $y : \mathscr{C} \to \mathscr{S}ets^{\mathscr{C}^{op}}$, then a sieve is a *subobject* $S \subset y(C)$ in the category of presheaves $\mathscr{S}ets^{\mathscr{C}^{op}}$.

9.2. *Remark.* Similar relative structures are actually employed in the case of Cohen's procedure as discussed in the *Being and Event*. The so called 'correct sets' $\female \in \mathscr{C}$ is similarly closed: whenever $p \in \female$ and $q \leq p$, then $q \in \female$.

If such 'multiple' ♀ is 'generic' — that is, it intersects every domination — it 'covers' the whole space.

9.3. **Definition.** A sieve S on C is called *closed* if and only if for all arrows $f : D \to C$, the pull-back $f^*M \in J(D)$ implies $f \in M$, which in turn implies that f^*M is the maximal sieve on D.

9.4. **Definition** (Subobject classifier)**.** For a Grothendieck-site $\mathscr{S}hvs(\mathscr{C}, J)$, there can be defined a subobject-classifier as

$$\Omega(C) = \text{ the set of closed sieves on } C,$$

which, in fact, is not only a presheaf but a sheaf since the condition of closedness in the category of presheaves, Ω could be defined to consist of all sieves.

9.5. *Remark.* In fact, given a sieve S on C, one may define a closure of S as[55]

$$\bar{S} = \{h \mid h \text{ has a codomain } C, \text{ and } S \text{ covers } h\}.$$

.

The closure operation is actually distinctive to any topos-theoretically defined notion of topology. In general, one defines topology in respect to the subobject-classifier Ω as an arrow $j : \Omega \to \Omega$ with $j^2 = j$, $j \circ \text{true} = \text{true}$ and $j \circ \wedge = \wedge \circ (j \times j)$, where the morphism true is a unique map $1 \to \Omega$ related to the definition of an elementary topos (see definition 10.3).

As I pointed out above, Badiou[56] himself defines a Grothendieck-topology when he demonstrates the existence of the 'transcendental functor'[57], that is, the sheaf-object. Badiou[58] defines the '*territory* of p' as

$$K(p) = \{\Theta \mid \Theta \subset T \text{ and } p = \Sigma\Theta\}.$$

They form a basis of a Grothendieck-topology. In short, every territory gives a sieve if it is completed under $\leq$-relation of T.

9.6. *Remark.* To formalise the compatibility conditions that Badiou's proof required, let me define these in terms of a Grothendieck-topology. A sieve S is called a *cover* of an object C if $S \in J(C)$. Let $P : \mathscr{C}^{op} \to \mathscr{S}ets$ be any presheaf (functor). Then a *matching family* for S of elements of P is a function which for each arrow $f : D \to C$ in S assigns an element $x_f \in P(D)$ such that $x_f \circ g = x_{fg}$ for all $g : E \to D$. The so called *amalgamation* of such a matching family is an element $x \in P(C)$ for which $x \circ f = x_f$. In Badiou's terminology, such matching families are regarded as pairwise compatible subsets of an object. An amalgamation — a 'real' element incorporating an atom does not need to exist. P is defined as *a sheaf* when they do so uniquely for all $C \in \mathscr{C}$ and sieves $S \in J(C)$. In

[55]Mac Lane & Moerdijk 1992, 141, Lemma 1.

[56]Badiou, *Logics of Worlds*, 2009. pp. 289–295

[57]In fact, there is an implicit definition of a Grothendieck-topology involved already when Badiou expresses the Cohen's procedure in the *Being and Event*

[58]Ibid. p. 291

Badiou's example, a matching family $\{x_f \mid f \in \Theta \subset T\}$ corresponds to an atom

$$\pi(a) = \Sigma\{\mathbf{Id}(a, x_f) \mid f \in \Theta\},$$

and this is precisely what I established in the above proof of the sheaf-condition[59]. The topology with a basis formed by 'territories' on p is, in fact, subcanonical, as in the category of the elements of T and arrows as order-relations. This is because for any $p \in T$ one has $\mathrm{Hom}_T(q, p) = \{f_{qp} \mid q \leq p\}$, and therefore as a set $h_p = \{q \mid q \leq p\}$ becomes canonically indexed by T. This makes $\mathfrak{h}_p$ bijective to a subset of T. Now for any territory Θ on p, a matching family of such subsets h_q are obviously restrictions of h_p. In general, for a presheaf P there can be constructed another presheaf possible to complete as a sheaf — sheafify (see remark 6.9) — and which is thus separated in the sense that amalgamations, if they exist, are unique. In practical calculations the fact that sheaves can always be replaced by presheaves and vice versa in the case of Grothendieck-sites is extremely useful.

9.7. *Remark.* As I have now discussed the more elementary designation of Grothendieck-topoi as categories of sheaves of sets, if one begins with the definition of the so called *elementary topos* defined merely as a category without any reference to $\mathscr{S}ets$, the stakes turned around (see definition 10.3 and remark 10.4).

Given these conditions, a topos is called Grothendieck and it retains a geometric morphism $\mathscr{E} \to \mathscr{S}ets$ related to these sheaf-representations even if the sheaf-functor does not have to result with an external Heyting algebra structure such as T. This only occurs if the morphism $\mathscr{E} \to \mathscr{S}ets$ is bounded and thus logical. More generally, when working over the 2-category of elementary topoi $\mathfrak{Top}$, if $f : \mathscr{E} \to \mathscr{F}$ is a geometric morphism which is bounded, there is an inclusion $\mathscr{E} \hookrightarrow \mathscr{F}^{\mathbf{C}}$, where $\mathbf{C}$ is an 'internal category' of $\mathscr{F}$. This follows from the so called Giraud–Mitchell–Diaconescu-theorem. Unless the geometric morphism $\gamma : \mathscr{E} \to \mathscr{S}ets$ is bounded, the $\gamma^*(\Omega_{\mathscr{E}})$ does not need to be a complete Heyting algebra. Only in the bounded, 'strong' case $\gamma_*(\Omega_{\mathscr{E}})$ is an external complete Heyting algebra and it is then the full subcategory of open objects in $\mathscr{E}$[60].

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[59] P is a sheaf, if and only if for every covering sieve S on C the inclusion $S \hookrightarrow h_C$ into the presheaf $h_C : D \mapsto \mathrm{Hom}(D, C)$ induces an isomorphism $\mathrm{Hom}(S, P) \cong \mathrm{Hom}(h_C, P)$. In categorical terms, a presheaf P is a sheaf for a topology J, if and only if for any cover $\{f_i : C_i \to C \mid i \in I\} \in K(C)$ in the basis K the diagram

$$P(C) \to \prod_i P(C_i) \rightrightarrows \prod_{i,j \in I} P(C_i \times_C C_j)$$

is an equaliser of sets (where the 'fibre-product' on the right hand side is the canonical pull-back of f_i and f_j (Mac Lane & Moerdijk 1992, 123). Furthermore, a topology J is called *subcanonical* if all representable presheaves, that is, all functors $h_C : D \mapsto \mathrm{Hom}(D, C)$ are sheaves on J.

[60] Ibid., 150.

10. Towards the Categorical Designation of Topoi

One needs to still close up the discussion regarding Badiou's mathematically materialist discourse by considering the relations between objects directly without referring to what the supposed objects 'consist of'. Thus I need to consider arrows (relations) between objects that are supposed to result with *natural transformations* (see 6.3) between sheaves. It is, indeed, the case that historically Grothendieck-topoi inspired the more abstract definition of an elementary topos defined in categorical language directly without any reference to the 'Platonic' category $\mathscr{S}ets$.

To proceed in the direction of abstract category theory, one should still demonstrate that Badiou's world of T-sets is actually the *category* of sheaves $\mathscr{S}hvs(T,J)$. To shift to the categorical setting, one first needs to define a relation between objects. These relations, the so called 'natural transformations', should satisfy conditions Badiou regards as 'complex arrangements'.

10.1. **Definition** (Relation). A *relation* from the object $(A, \mathbf{Id}_\alpha)$ to the object $(B, \mathbf{Id}_\beta)$ is a map $\rho : A \to B$ such that

$$\mathbf{E}_\beta\ \rho(a) = \mathbf{E}_\alpha\ a \quad \text{and} \quad \rho(a \between p) = \rho(a) \between p.$$

10.2. **Proposition.** *It is a rather easy consequence of these two presuppositions that it respects the order relation $\leq$ one retains*

$$\mathbf{Id}_\alpha(a,b) \leq \mathbf{Id}_\beta(\rho(a), \rho(b))$$

and that if $a \ddagger b$ are two compatible elements, then also $\rho(a) \ddagger \rho(b)$[61]*. Thus such a relation itself is compatible with the underlying T-structures.*

10.3. **Definition** (Elementary Topos). An *elementary topos* $\mathscr{E}$ is a category which

1) has finite limits, or equivalently $\mathscr{E}$ has so called pull-backs and a terminal object 1[62],
2) is Cartesian closed, which means that for each object X there is an *exponential functor* $(-)^X : \mathscr{E} \to \mathscr{E}$ which is right adjoint to the functor $(-) \times X$[63], and finally
3) (axiom of truth) $\mathscr{E}$ retains an object called the *subobject classifier* Ω, which is equipped with an arrow $1 \xrightarrow{\text{true}} \Omega$ such that for each monomorphism $\sigma : Y \hookrightarrow X$ in $\mathscr{E}$, there is a unique *classifying map*

[61] Badiou, *Logics of Worlds*, 2009. pp. 338–339.

[62] A terminal object 1 means that for every object X there is a *unique* arrow $X \to 1$, every arrow can be extended to terminate at 1. To the notion of a pull-back (or fibered product) we will come to shortly.

[63] Generally a functor between categories $f : \mathscr{F} \to \mathscr{G}$ maps each object $X \in \mathscr{F}$ to an object $f(X) \in \mathscr{G}$ and each arrow $a : X \to Y$ to an arrow $f(a) : f(X) \to f(Y)$ 'naturally' in the sense that $f(a \circ b) = f(a) \circ f(b)$. Therefore, f induces a map $f : \mathrm{Hom}_{\mathscr{F}}(X,Y) \to \mathrm{Hom}_{\mathscr{G}}(f(X), f(Y))$. If $g : \mathscr{G} \to \mathscr{F}$ is another functor, then $g \circ f : \mathscr{E} \to \mathscr{E}$ and $f \circ g : \mathscr{E} \to \mathscr{E}$ are two functors. It is said that another functor $g : \mathscr{G} \to \mathscr{F}$ is right (resp. left) adjoint of f, notated by $f \dashv g$, if given any pair of objects $X \in \mathscr{E}$ and $Y' \in \mathscr{G}$ there is also a 'natural' isomorphism $\Phi_{X,Y'} : \mathrm{Hom}_{\mathscr{F}}(X, g(Y')) \tilde{\to} \mathrm{Hom}_{\mathscr{G}}(f(X), Y')$.

$\varphi_\sigma : X \to \Omega$ making $\sigma : Y \hookrightarrow X$ a pull-back[64] of φ_σ along the arrow true.

10.4. *Remark* (Grothendieck-topos). In respect to this categorical definition, a Grothendieck-topos is a topos with the following conditions (J. Giraud)[65] satisfies: (1) $\mathscr{E}$ has all set-indexed coproducts, and they are disjoint and universal, (2) equivalence relations in $\mathscr{E}$ have universal coequalisers. (3) every equivalence relation in $\mathscr{E}$ is effective, and every epimorphism in $\mathscr{E}$ is a coequaliser, (4) $\mathscr{E}$ has 'small hom-sets', i.e. for any two objects X, Y, the morphisms of $\mathscr{E}$ from X to Y are parametrized by a set, and finally (5) $\mathscr{E}$ has a set of generators (not necessarily monic in respect to 1 as in the case of locales). Together the five conditions can be taken as an alternative definition of a Grothendieck-topos (compare to definition 6.5).

11. Badiou's Struggle with 'Diagrammatics'

Regardless of Badiou's[66] confusion about the structure of the 'power-object', it is safe to assume that Badiou has demonstrated that there is at least a *category* of T-$\mathscr{S}ets$ if not yet a topos. Its objects are defined as T-sets situated in the 'world $\mathbf{m}$'[67] together with their respective equalisation functions $\mathbf{Id}_\alpha$. It is obviously Badiou's 'diagrammatic' aim is to demonstrate that this category is a topos; and ultimately to reduce any 'diagrammatic' claim of 'democratic materialism' to the constituted, non-diagrammatic objects such as T-sets. In this section, I discuss Badiou's 'struggles' regarding this affair. The 'struggle' relates not only to mathematics as a technical quality but implies philosophical inaccuracies as well.

Confusing the generality of topoi with the specificity of locales, he believes to surmount the need to work categorically, through 'diagrams', and to reach again that 'Platonic' plane of 'objects' which Kant's limited imperative of objectivity prohibited. Thus he believes to revert topos theory and category theory back to set theory and ultimately denounce this new, 'post-structuralist' sphere of power which operates 'diagrammatically'. That is, he inevidently refers to that notion of 'diagrams' of power that became utilised by many 'post-structuralist' after it was first launched by Michel Foucault[68].

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[64]In this particular case, the pull-back-condition means that, given any $\theta : Z \to X$ so that $\varphi_\sigma \circ \theta : Z \to \Omega$ is the (unique) arrow $Z \longrightarrow 1 \xrightarrow{\text{true}} \Omega$, then there is a unique map $\gamma : Z \to Y$ so that $\sigma \circ \gamma = \theta$.

[65]For the definition and further implications see Johnstone, *Topos Theory*, 1977, *Topos Theory*, pp. 16–17.

[66]Badiou, *Logics of Worlds*, 2009. p. 339.

[67]Despite the fact that the category of $\mathscr{S}ets$ — which isn't itself a set as Badiou demonstrates in the case of the 'world $\mathbf{m}$' — is 'paradoxic' given the Russell's paradox, one may now safely replace the category $\mathscr{S}ets$ with any category $\mathscr{S}$, which satisfies the basic properties of the set-theoretic world $\mathscr{S}ets$ — it satisfies a categorical version of the axiom of choice (AC) and (SG) and has a natural number object; and finally its subobject-classifier equals $\mathbf{2}$ — it is 'bi-valued'.

[68]Foucault, Michel (1995), *Discipline and Punish: The Birth of the Prison*, trans. Alan Sheridan. New York: Random House. pp. 171, 205.

Instead of *mathematically reducing* diagrammatics back to the more 'Platonic' edifice of ontology, Badiou himself turns out to be reductionist at least when it comes to his naïve mathematical understanding of diagrammatics — the categorical *modus operandi.* This reductionism is highlighted by the elementary mistakes Badiou's own engagement with categorical 'diagrammatics' demonstrates.

To discuss these mistakes, when it comes to the actual 'mathemes' in this category of T-$\mathscr{S}ets$, he demonstrates that there exists an object which he calls an 'exponent' of a relation $X \to Y$. In other words, '[i]f we consider [...] two Quebecois in turn as objects of the world, we see that they each entertain a relation to each of the objects linked together by the 'Oka incident' [...] and, by the same token, a relation to this link itself, that is a relation to a relation'[69]. In diagrammatic terms, Badiou[70] defines a relation to be "universally exposed' if given two distinct expositions of the same relation, there exists between the two exponents [sic] one and only one relation such that the diagram remains commutative', and for that he draws the following diagram:

$$\begin{tikzcd}
 & \text{Exponent 1} \arrow[dl] \arrow[drr] & \text{Exponent 2} \arrow[l, "\text{unique relation}"'] \arrow[dll] \arrow[dr] & \\
\text{Object 1} \arrow[rrr, "\text{Exposed relation}"] & & & \text{Object 2.}
\end{tikzcd}$$

On the basis of this diagram, Badiou[71] argues that if the previous diagram with the 'progressive Qubebecois citizen' (Exponent 1) was similarly exposed also with another, 'progressive citizen' (Exponent 2), then 'if the reactionary supports the progressive, who in turn supports the Mohawks, there follows a flagrant contradiction with the direct relation of vituperation that the reactionary entertains with the Mohawks'. The claims is unwarranted. Of course, particular relations involved determine the commutativity of the diagram, but Badiou *falsely argues for the necessity of its non-commutativity.*

11.1. **Proposition.** *Badiou's condition of universal exposition is falsely stated. Given* any relation $f : X \to Y$*, it is easy to construct an object that does not satisfy Badiou's universality condition in almost any topos, in particular in* $\mathscr{S}ets$*.*

Proof. For illustration, let me assume that $T = \{\text{false} < \text{true}\}$ so that I am actually in the category of $\mathscr{S}ets$ and can work on 'elements' point-wise. For example, I can consider the diagram with π_1, π_2 denoting the two projection

[69] Ibid., 313.
[70] Ibid., 317
[71] Ibid., 315–316.

maps of the Cartesian product $X \times X \to X$,

$$\begin{tikzcd}
X \times X \times X \arrow[rr, dashed, "h?"] \arrow[d, "\pi_1"'] \arrow[drr, "f \circ \pi_1" near end] & & X \times X \times X \arrow[d, "f \circ \pi_2"] \arrow[dll, "\pi_2"' near end] \\
X \arrow[rr, "f"] & & Y.
\end{tikzcd}$$

It is clear that the third component of the map h — namely the map $\pi_3 \circ h \circ \iota_3$ — can be defined arbitrarily because in any of the maps with the domain $X \times X \times X$ there are no references to the third projection-component. The only condition concerns the first component of h: it has to satisfy specific conditions with respect to the second component in the image of the map h. In other words, in Badiou's definition a relation is never universally exposed as long as there is enough room to play such as in the case of the category of $\mathscr{S}ets$.

There is a further terminological confusion. This regards to how Badiou calls the object to 'expose' a relation. As such his 'exponent' obviously contradicts with the standard definition of category theory, considering it as the object X^Y of *all* relations between X and Y instead of exposing only a particular relation, say $f : X \to Y$. $\square$

In short, Badiou's exponents are *graphs* in standard mathematical terminology. The correct form of the exposition — to maintain the connections Badiou makes with the case of Quebec — would then be that a relation is universally exposed by *a particular object* Γ_f so that whenever another object Z also exposes the relation, there is a unique arrow $Z \to \Gamma_f$ which makes the diagram to commute. What is even more severe, however, is the mistake to forget to designate that particular object Γ_f (F_ρ in Badiou's example), *up to isomporphism* — to specify what makes it structurally *unique*. It is this unique isomorphism class generally conditioned for an object which should satisfy a so called *universal property* as otherwise the property would not be strong enough to specify a categorical structure of an object. The universal exposition of the relation f is not just a property of the relation but the particular 'universal object' in respect to that relation f; in the above example it defines not any possible citizen but a unique form of a 'universal citizen' associated with the particular relation of the 'Oka incident'; call it $\Gamma_{\text{Oka incident}}$.

11.2. *Remark.* This flaw in defining what precisely is 'universal' in the object specified by the universal condition demonstrates that Badiou's understanding of what is specific to category theory is *deficient.* In other words, he fails to acclaim precisely that 'Kantian' categorical shift that doesn't specify objects directly, as what they *are*, but diagrammatically, in regard to how they *relate* to other structures. In fact, similar 'universal properties' were typical to modern mathematics including branches such as homological algebra and topology even before category theory formally developed. Category

theory makes such properties one of the *key operatives* of mathematical 'diagrammatics' without which contemporary mathematics would be impossible. Badiou thus rather violently crosses that Kantian limit when enforcing the set-theoretic premise of local topos theory. Given his enforced condition any generality of the arguments Badiou could hold against (post-)structuralist diagrammatics is thus mathematically falsified in advance.

For example, the first condition of the general definition of an elementary topos $\mathscr{E}$ requires there to exist finite limits[72] but it is equivalent to there being the so called terminal object and pull-backs. For any given two arrows $f : X \to Z$ and $g : Y \to Z$, a pull-back of those is an object $X \times_Z Y$ such that the inner diagram commutes and if satisfies a universal condition that if the whole diagram commutes, every arrow $u : W \to X \times_Z Y$ is unique up to an isomorphism:

$$\begin{tikzcd}
W \arrow[dr, dotted, "u"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r, "p"'] \arrow[d, "q"] & X \arrow[d, "f"'] \\
& Y \arrow[r, "g"] & Z.
\end{tikzcd}$$

11.3. **Proposition** (Exposition of a Singler Relation)**.** *The corrected version of Badiou's universal 'exposition'* of a particular relation *now comes back to saying that there exists a pull-back*

$$\begin{tikzcd}
F_\rho = \Gamma_{\rho : A \to B} \arrow[r, dashed, "f"] \arrow[d, dashed, "g"] & X \arrow[d, "\rho"'] \\
B \arrow[r, equal, "1_B"] & B
\end{tikzcd} \tag{1}$$

in the category of T-$\mathscr{S}ets$.

11.4. *Remark.* With only a few more complications Badiou' proof of the above proposition could easily be transformed into showing that *any* pull-back exists. This would, in fact, complete the proof of the first condition of an elementary topos (definition 10.3). This may follow a route similar to the demonstration of the existence of a graph which Badiou, despite the misplaced definition, accomplishes.

Partial proof. Badiou phrases the universal condition mistakenly and doesn't account to its *uniqueness*. He demonstrates only the *existence* of such a

[72]If a category $\mathfrak{J}$ is finite (in the number of objects and arrows), I can consider the category $\mathscr{E}^{\mathfrak{J}}$ whose objects are functors $F : \mathfrak{J} \to \mathscr{E}$ and arrows so called 'natural transformations'. There is a diagonal functor $\Delta_{\mathfrak{J}} : \mathscr{E} \to \mathscr{E}^{\mathfrak{J}}$ defined as the constant functor associating $\Delta(X)(j) = X$ for all $j \in \mathfrak{J}$. Then the limit-condition means that there is a right adjoint to $\Delta_{\mathfrak{J}}$ which is $\varprojlim_{\mathfrak{J}} : \mathscr{E}^{\mathfrak{J}} \to \mathscr{E}$ and designates a limit of $\mathfrak{J}$-indexed objects. An alternative definition supposes only the existence of pull-backs and a terminal-objects from which the existence of all finite limits follows. Those are limits of the categories of the empty category (terminal object) and the category with only one object and two nonidentity morphisms $\to \bullet \leftarrow$ (pull-back).

graph whereas uniqueness would need a correction to the way in which the universality property is stated. The proof follows by first demonstrating that normal Cartesian product exists: '[g]iven two multiples A and B appearing in a world, the product of these two sets, that is the set constituted by all the ordered pairs of elements of A and B (in this order), must also appear in this world'[73]. The next step consists of showing for a relation $\rho : (A, \mathbf{Id}_\alpha) \to (B, \mathbf{Id}_\beta)$ that the multiple consisting of pairs $(x, \rho(x)) \subset A \times B$, denoted by F_ρ is a multiple itself[74], and if equipped with a map $\nu : F_\rho \to T$, where

$$\nu((a, \rho(a)), (b, \rho(b))) = \mathbf{Id}_\alpha(a, b) \wedge (\rho(a), \rho(a')),$$

this map satisfies the conditions of a transcendental indexing[75]

$$\nu(x, y) \wedge \nu(y, z) \leq \nu(x, z).$$

To show that F_ρ is an object, one is thus required to show that '[e]very atom is real'[76]. Given an atom $\epsilon : F_\rho \to T$ Badiou[77] constructs a map $\epsilon^* : a \mapsto \epsilon(a, \rho(a)) : A \to T$, which is an atom since ϵ satisfies the corresponding conditions. Hence by the 'postulate of materialism' it is real; say $\epsilon^*(x) = \mathbf{Id}_\alpha(c, x)$. But now $\epsilon(x, \rho(x)) = \nu[(c, \rho(c)), (x, \rho(x))]$ which proves Badiou's Lemma 4. By his Lemma 5, Badiou then demonstrates that the diagram (1) is valid: '[t]he object (F_ρ, ν) is an exponent of the relation ρ', which follows by showing that 'f and g conserve localizations', ie. $f(a \wr p) = f(a) \wr p$ and $g(a \wr p) = g(a) \wr p$; and that $(a, \rho(a)) \wr p = a \wr p$. Because of the definition of $\mathcal{F}_p$ it is easy to set-theoretically see that it satisfies the universality condition of the pull-back diagram (1) and thus Badiou[78] is left to show only that it similarly 'conserves existences and localizations'.

$\square$

12. T-sets Form a Topos — A Corrected Proof

I need to close the discussion regarding the 'logical completeness of the world' raised by Badiou[79], given his failure to verify the conditions of a topos.

12.1. **Theorem.** *Badiou's world consisting of T-$\mathscr{S}ets$ — in other words pairs $(A, \mathbf{Id})$ where $\mathbf{Id} : A \times A \to T$ satisfies the particular conditions in respect to the complete Heyting algebra structure of T — is logically closed in the sense that it is a topos. It thus retains not only the pull-back-expositions of its relations but also the exponential functor, the subobject classifier and the power functor, which makes a topos-theoretic internalisation of Badiou's infinity arguments possible.*

[73]Badiou, *Logics of Worlds*, 2009. p. 345.
[74]Ibid., lemma 2, 346.
[75]Ibid., lemma 3, 346.
[76]Ibid., lemma 4, 347.
[77]Ibid., 347–348.
[78]Ibid., 350–352.
[79]Ibid., 318.

Proof. I need to demonstrate that Badiou's world is a topos. Rather than beginning from Badiou's formalism of T-sets, I refer to the standard mathematical literature based on which T-sets can be regarded as sheaves over the particular Grothendieck-topology on the category T: there is a categorical equivalence between T-sets satisfying the 'postulate of materialism' and $\mathscr{S}hvs(T, J)$. The above complications regarding the 'concrete' demonstration on the level of the 'Platonic' multiples constituting T-sets thus goes in vain. I only need to show that $\mathscr{S}hvs(T, J)$ is a topos.

Consider the adjoint sheaf functor that always exist given a category of presheaves

$$\mathbf{Id}_\alpha : \mathscr{S}ets^{\mathscr{C}^{op}} \to \mathscr{S}hvs(\mathscr{C}^{op}, J),$$

where J is the canonical topology. It amounts to an equivalence of categories. Thus it suffices to replace this category by the one consisting of presheaves $\mathscr{S}ets^{T^{op}}$. This argument works for any category $\mathscr{C}$ rather than the specific category related to an external complete Heyting algebra T. In the category of $\mathscr{S}ets$ define Y^X as the set of functions $X \to Y$. Then in the category of presheaves $\mathscr{S}ets^{\mathscr{C}^{op}}$

$$Y^X(U) \cong \mathrm{Hom}(h_U, Y^X) \cong \mathrm{Hom}(h_U \times X, Y),$$

where h_U is the representable sheaf $h_U(V) = \mathrm{Hom}(V, U)$. The adjunction on the right side needs to be shown to exist for all sheaves; not just representable ones. The proof then follows by an argument based on categorically defined limits whose existence is almost a trivial task[80]. It can be also verified directly that the presheaf Y^X is actually a sheaf.

Finally, for the existence of the subobject-classifier $\Omega_{\mathscr{S}ets^{\mathscr{C}^{op}}}$[81], it can be defined as

$$\Omega_{\mathscr{S}ets^{\mathscr{C}^{op}}}(U) \cong \mathrm{Hom}(h_U, \Omega) \cong \{\text{sub-presheaves of } h_U\} \cong \{\text{sieves on U}\},$$

or alternatively for the category of proper sheaves $\mathscr{S}hvs(\mathscr{C}, J)$,

$$\Omega_{\mathscr{S}hvs(\mathscr{C},J)}(U) = \{\text{closed sieves on U}\}.$$

Here it is worth reminding that the topology on T was defined by a basis $K(p) = \{\Theta \subset T \mid \Sigma\Theta = p\}$. Therefore, in the case of T-sets satisfying the strong 'postulate of materialism', $\Omega(p)$ consists of all sieves S (downward dense subsets) of T bounded by relation $\Sigma S \leq p$. These sieves are further required to be *closed*. A sieve S with envelope $\Sigma S = s$ is *closed* if for any other $r \leq s$, ie. for all $r \leq s$, one has the implication

$$f^*_{rs}(S) \in J(r) \quad \Longrightarrow \quad f_{rs} \in S,$$

where $f_{rs} : r \to s$ is the unique arrow in the poset category. In particular, since $\Sigma S = s$ for the topology whose basis consists of territories on s, I have the equation $1^*_s(S) = f^*_{ss}(S) = S \in J(s)$. Now the condition that the sieve is closed implies $1_s \in S$. This is only possible when S is the maximal sieve on s — namely it consists of *all* arrows $r \to s$ for $r \leq s$. In such a case it is

[80]Johnstone, *Topos Theory*, 1977, pp. 24–25.
[81]Ibid., p. 25.

easily verified that S itself is closed. Therefore, in this particular case

$$\Omega(p) = \{\downarrow (s) \mid s \leq p\} = \{h_s \mid s \leq p\}.$$

It is easy to verify that this is indeed a sheaf whose all amalgamations are 'real' in the sense of Badiou's postulate of materialism. Thus it retains a suitable T-structure.

Let me assume now that I am given an object A, which is basically a functor and thus a T-graded family of subsets $A(p)$. For there to exist a sub-functor $B \hookrightarrow A$ comes down to stating that $B(p) \subset A(p)$ for each $p \in T$. For each $q \leq p$, I also have an injection $B(q) \hookrightarrow B(p)$ compatible (through the subset-representation with respect to A) with the injections $A(q) \hookrightarrow B(q)$. For any given $x \in A(p)$, I can now consider the set

$$\varphi_p(x) = \{q \mid q \leq p \text{ and } x \between q \in B(q)\}.$$

This is a sieve on p because of the compatibility condition for injections, and it is furthermore closed since the map $x \mapsto \Sigma\varphi_p(x)$ is in fact an atom (exercise) and thus retains a real representative $b \in B$. Then it turns out that $\varphi_p(x) = \downarrow (\mathbf{E}b)$. I now possess a transformation of functors $\varphi : A \to \Omega$ which is natural (diagrammatically compatible). But in such a case I know that $B \hookrightarrow A$ is in turn the pull-back along φ of the arrow true:

$$\begin{tikzcd} B \arrow[r] \arrow[d, hook] & 1 \arrow[d, "\text{true}"] \\ A \arrow[r, "\varphi"] & \Omega. \end{tikzcd}$$

It is an easy exercise now to show that this map satisfies the universal condition of a pull-back. Indeed, let $\theta : A \to \Omega$ be another natural transformation making the diagram commute. Given $x \in A(p)$ and $q \to p$, the pull-back-condition means that $x \between q \in B(q)$ if and only if $\theta(x \between q) = \text{true}_q$, but by naturality of θ this is same as saying $\theta_p(x) \between q = \text{true}_q$. That in turn means $(q \to p) \in \theta_q(x)$. Therefore, as sets, $\varphi_q(x) = \theta_q(x)$. This concludes my sketch that true : $1 \hookrightarrow \Omega$ associating to each singleton of $1(p)$ the maximal sieve $\downarrow (p)$ makes Ω the subobject classifier of the category $\mathscr{S}ets^{T^{op}}$. This is equivalent to the category of T-$\mathscr{S}ets$. □

13. Badiou Inside Out: It's About Relations, Not Objects

In the fourth and the final book of the 'Greater Logic', that is, of the first part of the *Logic of Worlds*, Badiou attempts to accomplish the above proof constituting the 'analytic' of the inquiry 'bearing only on the transcendental laws of being there', or in other words 'the theory of worlds, the elucidation of most abstract laws of that which constitutes a world qua general form [sic] of appearing'. However, Badiou fails to make the required verifications in order to support the claim of the 'generality' of such a form[82]. Contra reaching such 'generality', Badiou reconciles this apparent contradiction inherent to his own edifice by adopting a *locally bounded*,

[82]Badiou, *Logics of Worlds*, 2009. p. 299.

reductive approach[83]. If Badiouian dialectics then aims to bridge the 'analytics' of the *Logic of Worlds* with the dialectics of the 'event' expressed in the *Being and Event*, the phrase 'analytic' refers exactly to the strong, initially *bounding* condition of constitution on which the *atomic analytics* of materialism of the *Logic of Worlds* is reduced rather than reaching the broader framework of 'weak materialism' that would allow materially non-representable torsion to subsist. As I pointed out, Grothendieck's *trade-off* with which approached such globally 'torsional' objects with only *locally effective hierarchical structures* — sieves — is *'synthetic' and inscriptive* rather than 'analytic' and incorporeal. In contrast, Badiou's 'analytics' of the 'unity' constitutive to 'real synthesis' is based on the *unnecessary supposition* of the unity of the postulate of materialism itself. This postulate itself is a split postulate which can be unified only on the *quasi-split*[84] situation of locales.

This seems to contradict Badiou's other maxim: if dialectics understands 'truths as exceptions', these exceptions need to be understood in terms of the 'most abstract laws' as the 'objective domain of their emergence' which 'cannot yet attain the comprehension of the terms that singularize materialist dialectic [...] and subjects as the active forms of these exceptions'[85]. Badiou's struggle with 'diagrammatics' makes his own understanding of these 'abstract laws' *flawed*. Indeed, Badiou's own understanding of materialist truth becomes thus *exceptional* based either on the primacy of the split concept of truth (the axiom of choice in the *Being and Event*) or the *quasi-split* materialisation forcing out any global torsion in fidelity with Badiou's atorsional *law of the subject*[86]. Instead, not only objects, as they appear, but their relations should form a crucial part of this analytic; Badiou's own dialectics, however, seems to be dominated by the bounded, set-theoretic representation of objects.

In short, Badiou fails to attest his own dialectic maxims. Badiou's 'object exhausts the dialectic of being and existence' that to Plato consisted of 'univesal part' in 'their participation in the Idea' and thus in Badiou's reading as 'pure, mathematically thinkable multiplicit[ies] that underl[y]' them. Only sets are 'thinkable' to Badiou whereas to Kant the answer of objectivity 'is

[83]I showed how he reduces topos theory to a theory of so called locales — elementary topoi that are defined over the category of $\mathscr{S}ets$ in such a way that the geometric morphism $\gamma : \mathscr{E} \to \mathscr{S}ets$ is *bounded* and *logical.* This makes Badiou to falsely identify the 'internal' logic of a topos with the logic which emerges on the set-theoretic surface of $T := \gamma_*(\Omega)$ because for a general $\mathscr{S}ets$-topos (eg. Grothendieck-topos), these two logics do not agree.
[84]The split understanding of truth refers to the condition of the category of $\mathscr{S}ets$ in which the subobject-classifier Ω is the split set consisting of only two elements: true and false. Similarly, although the truth in a local doesn't split in this explicit manner, it is quasi-split inasmuch as the T anyway set-theoretically represents truth as extensive 'values' or 'degrees'. As a remark, Badiou misleadingly regards them as 'degrees of intensity' inasmuch as the categorical designation of a topos expresses such 'degrees' in a much more intensive meaning. In contrast, Badiou's set-theoretic representation makes these 'degrees of truth' *extensive* when they are set-theoretically incorporated.
[85]Ibid. p. 299.
[86]Badiou, *Being and Event*, 2006. p. 401.

oriented towards universality as the subjective constitution of experience'[87]. The emphasis Badiou places on the *strong* postulate of materialism in his opposition towards 'democratic materialism' — vitalism in its disguise — becomes understandable only from his reluctance to give up 'Platonism' as the foundation of mathematics Badiou swears by in the name of set theory. Even if Badiou[88] confirms that '[t]he transcendental is not subjective, nor is it as such universal', Badiou falsely assumes the *form of transcendental* to be unique in respect to *materialism*. In other words, Badiou designates the transcendental as an ordered Heyting algebra *forcing* a topos to be logically bounded according to the 'Platonic' affirmation of the universality of set-theoretic ontology. That is a mistake his own philosophy cannot correct. Whereas the more 'Kantian' category theory draws its objectivity exactly from the fact that the boundaries of objects should not be crossed refusing to assume any incorporeal consistency, Badiou's 'analytics', in contrast, has *initially chosen* to regard set theory ($\mathscr{S}ets$) as the *only* foundation of objectivity.

It could be argued that Badiou might have made the above *mathematical mistakes* by accident. Locales in logic are particularly important as their internal logic is intuitionist and there is nothing wrong in working over locales as long as one is reminiscent of their specificity. Badiou himself hardly counts as such an 'intuitionist', however. As Badiou[89] convicts in the *Being and Event*, 'that intuitionism has mistaken the route in trying to apply back onto ontology criteria of connection which *come from elsewhere*, and especially from a doctrine of mentally effective operations'. In the *Logic of Worlds*, he becomes blind precisely to this categorical, 'diagrammatic' movement which, indeed, comes from elsewhere. The *Logic of Worlds* thus appears to contradict the *Being of Event* in which Badiou[90] declares that 'intuitionism is a prisoner of the empiricist and illusory representation of mathematical objects' and therefore emphasising not only the primacy of objects over relations (opposed to categorical theory): '[h]owever complex a mathematical proposition might be, if it is an affirmative proposition it comes down to declaring the existence of a pure form of the multiple'; thus not the diagrammatic *form of relations* but of the multiple itself. With the fundamental law of the subject substantialising the ordered, atorsional form of the transcendental, Badiou thus seems to have fallen into his own trap exactly by designating the consistent anatomy of objects as T-sets whose consistence has already been predetermined in the first place — not as an effective result of the way in which objects are composed within the diagrammatic relations in respect to each other. This claim seems to be further supported by the fact that it is precisely the more 'Grothendickian', inscriptive sieve-structures imposed on set-theoretic constructions synthetically that gave rise to Cohen's original strategy which became a central

[87]Ibid., p. 301.
[88]Ibid. p. 301.
[89]Badiou, *Being and Event*, 2006. p. 249.
[90]Ibid. p. 249.

cornerstone of Badiou's subject-philosophy in the *Being and Event* contra the reductivism of the *Logic of Worlds*.

Indeed, Badiou's discussion on intuitionism supports the claim that Badiou's mathematical mistakes are unfortunate accidents making the strong hypothesis of materialism only a coincidental oversight. However, the evidence supporting the accident hypothesis is only circumstantial. That hypothesis would be based on the assumption that only Badiou's mathematical reasoning is flawed, based on his reduced understanding of a specific branch of local topos theory[91]. The philosophical implications, in contrast, support the claim that such an 'accident' occurs philosophically on purpose.

Indeed, Badiou[92], consequentially, claims to be able to contribute to several philosophical debates on the basis of these inadequate premises. The first problem he discusses has been haunting philosophy since the atomists Democritus to Lucretius, who 'had glimpsed the possibility of an infinite plurality of worlds'. Especially this concerns the capacity of set theory to reflect upon infinity — the possibility of 'void' ascribing infinity to being-there. As Badiou[93] continues, to Aristotle the question of the 'arrangement of the world [which] is essentially finite' he claims that the problem persisting 'at the heart of the Kantian dialectic'. Even if Badiou is right about the problem which persists, one should question whether the ultimate focus should be put on the initial object 'void' (in $\mathscr{S}ets$) rather than on the *category theoretically relevant, terminal object One* in ascribing — or rather prescribing the regime of infinity which the world (topos) can grasp. From the topos-theoretic perspective the first question — the void ascribing infinity — should probably be displaced by the contrary question in which the One terminates the worldly finitude whilst simultaneously reflecting the infinity of the world: the One itself could retain an infinite number of subobjects. This falsifies Badiou's argument against Leibniz's 'constructivism' which grounds on Badiou's reduced 'analytics' rather than Leibniz — and other 'mysticists' — themselves.

Yet even such categorical diagrammatics cannot grasp the abstract infinitude of geometry. This relates to the second problem raised up by Kant. Badiou[94] argues the categorical objects to be prescribed with their explanative and legislative primacy: an 'object is nothing but the legislation of appearing'. But this is *not the same* thing as stating: '[t]he definition of a relation must be strictly dependent on that of objects', which makes Badiou owe 'Wittgenstein who, having defined the 'state of affairs' as a 'combination of objects', posits that, 'if a thing can occur in a state of affairs,

[91] In the mathematical references Badiou makes, everything seems to relate only to such logically bounded, localic structures. Badiou (ibid. p. 538) mentions such books as:
Bell, *Toposes and Local Set Theories: An Introduction*, 1988.
Borceux, *Handbook of Categorical Algebra. Basic Theory. Vol. I.*, 1994.
Goldblatt, *The Categorical Analysis of Logic*, 1984.
Wyler, *Lecture Notes on Topoi and Quasi-Topoi*, 1991.
[92] Badiou, *Logics of Worlds*, 2009. p. 300.
[93] Ibid. p. 300
[94] Ibid. p. 301

the possibility of the state of affairs must be written into the thing itself". Contra arguing that Badiou would have solved the Kantian problematics of analytics and its synthesis, what if, instead, the state ascribes *things into the writing (of relations) itself.* Such contrary position would seem to be in fidelity with Grothendieck's *inscriptive* account on neighborhoods which is opposite to what Wittgenstein aimed to say. The thing, geometric object, is ascribed into writing, that is, geometry is ascribed to the hierarchical sieves grasping the 'inscriptive' neighborhoods. This is the opposite to saying that writing would be 'written into the thing itself', that is, that these ascribed, hierarchical constructs would be written into geometry. Topos theory takes an anti-Wittgensteinian position whilst only Badiou's 'Platonic' guilt dilutes such a vision.

Thus topos theory can reflect upon the Kantian dialectics of relativism abstractly and conceptually much better than Badiou's confined theory of locales. Even if the world couldn't conceptualise its own infinity, it can grasp the consistent 'interior of the world'[95], insofar as it is effected by language, diagrammatically into which the thing — the object — is synthetically ascribed. Topos theory thus counts exactly as an 'expansive construction of the multiple to the multiple-beings'[96] when they are not required to be 'transcendentally indexed' in the extensive, Badiouian sense of the poset-order. It turns out that set-theoretic framework which Badiou takes as the absolute limit of mathematical ontology is actually accessible by internal operators of the world when the world is interpreted as a topos. This directly contradicts with what Badiou believes because the limits of mathematics and its relative grasp of the infinite are not anymore impossible to grasp to those 'who exist in the world, or who enjoys a non-nil self-identity'[97].

As a final remark in respect to Badiou's first dialectics of the event, the event is defined negatively according to the mathematical regulation through which its consequences are intervened. It would be interesting to contrast topos theory — as a real mathematical change — to his own event philosophy. Badiou's fidelity in set theory makes him blind to precisely this new, topos-theoretic frontier that withholds Cohen's procedure by other, 'diagrammatic' means. It is *another* mathematical *grammar* that doesn't designate the analytic plane of causation but *internalises* the 'dialectic impasses' pertinent to such structural 'torsion' that obstructs morphisms and ultimately the mathematical truth itself from being *split.* It is an alternative, inscriptive approach to mathematics that seems to falsify at least one of Badiou's two arguments: (1) that the Event cannot be approached mathematically except by regulating its consequences; and (2) that formal logic is the only meta-structural foundation of such regulatory intervention of truth. Any one of these two propositions can survive only on the detriment of the other. Such subjective 'torsion' discerned by Cohen's procedure can, in fact, be regulated by the means of category theory. If, in contrast,

[95]Ibid. p. 262.
[96]Ibid. p. 309.
[97]See ibid. p. 310.

one denounces the mathematical status of category theory and reduces it to the way in which it is subjugated to set-theoretic ontology, the categorical approach is non-mathematical *stricto sensu* and thus fails to regulate the event it inscribes. Given how category theory obstructs Badiou's subject-philosophy of the event, topos theory would in both cases turns out to be detrimental to Badiou's edifice.

14. A Remark on Mathematical Science Studies

It is clear that Badiou's originality lies in his attempt to bridge many of the gaps pertinent to contemporary divisions of philosophy and science, even if in this respect Badiou hasn't mastered his course in purely mathematical terms. Not only the philosopher Badiou, but Badiou *studying mathematics and its limits* seems to have opened a radically new perspective in *science studies*, a new style that appears to fulfill that Gilles Deleuze's[98] claim that style, 'in a great writer', 'is always a style of life'. Even if Badiou's mathematical personality makes the disparities of set-theoretical ontologism recur — thus making him convicted with many constructivist accounts on mathematics[99], Badiou eventually opens up a new style of life too, which is Badiou's real virtue 'inventing a possibility of life, a way of existing'. He, again, opens up the question of *experimenting* mathematics and philosophy; it doubles the mathematical play of forces in a new scientifico-philsophical self-relation. This, what I call the moment of 'mathematical science studies', happens even if Badiou is unaware of what his the intensity of his style may reach in combination with a more intense account on topos theory.

Historically it seems that mathematical language has developed together with philosophy — in the hands of Descartes, Leibniz, Spinoza, Hegel, Russell and many others. It is time to bridge that gap again. This is all the more important, given how the ethical domain of science was violently bursted between the 'post-structuralist' thinkers and scientists due to Alan Sokal's[100] hoax. The 'progress' of science isn't always as 'democratic' or 'orderly' as one would like to persue[101].

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[98]Deleuze, Gilles (1990), *Negotiations, 1972–1990.* Trans. Martin Joughin. New York: Columbia University Press. , p. 100.

[99]Bloor, David (1991), *Knowledge and Social Imagery.* Chicago: University of Chicago Press.

[100]See Sokal, Alan (1996), 'Transgressing the Boundaries: Toward a Transformative Hermeneutics of Quantum Gravity', *Social Text* 46/47, Science Wars, pp. 217–252.
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